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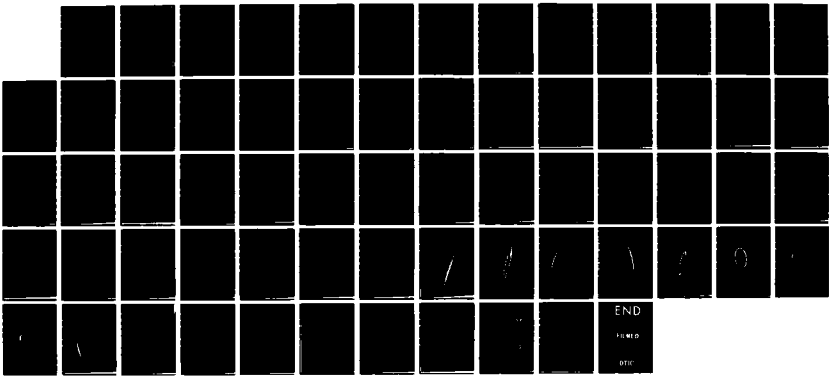

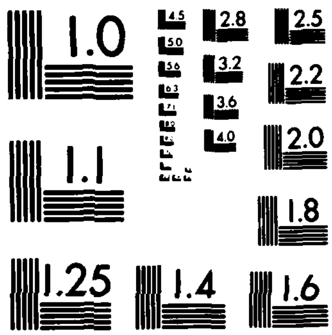

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The main aim of this research has been to develop and implement a technique for the generation of spatial coordinates in 3D regions enclosed by arbitrary smooth surfaces for ultimate use in the numerical solution of the Navier-Stokes equations. In this regard, a mathematical model based on a set of elliptic PDE's has been developed, which has been used to generate smooth coordinates in the region formed by arbitrary inner and outer surfaces of known shapes, around multibodies, particularly around a wing-body combination. These equations have also been used to generate surface coordinates in arbitrary surfaces and are also capable of coordinate redistribution in any desired manner both in 3D regions and in 2D surface regions.

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THE GENERATION OF THREE-DIMENSIONAL BODY-FITTED COORDINATE SYSTEMS FOR VISCOUS FLOW PROBLEMS

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The Generation of Three-Dimensional Body-Fitted Coordinate Systems for Viscous Flow Problems

by Z.U.A. Warsi*

Abstract

The main aim of this research has been to develop and implement a technique for the generation of spatial coordinates in 3D regions enclosed by arbitrary smooth surfaces for ultimate use in the numerical solution of the Navier-Stokes equations. In this regard, a mathematical model based on a set of elliptic PDE's has been developed, which has been used to generate smooth coordinates in the region formed by arbitrary inner and outer surfaces of known shapes, around multibodies, particularly around a wing-body combination. These equations have also been used to generate surface coordinates in arbitrary surfaces and are also capable of coordinate redistribution in any desired manner both in 3D regions and in 2D surface regions.

*Principal Investigator.

Introduction

The problem of numerical grid generation in surfaces and in three-dimensional configurations through elliptic PDE's (Part of [illegible]) has been pursued under this grant period. Various reports and journal publications produced under the present grant have explained the mathematical model in detail. (Refer to a list of publications produced in the grant period). The developed mathematical model has been programed on CRAY-1 and has been tested for single and two-body configurations enclosed in a single outer boundary (refer to Appendix C) and for generation of coordinates in a single surface (refer to Appendix B). The main elements of the mathematical development are contained in Appendix A.

Papers and Reports Produced Under the Grant

1. "A Method for the Generation of General Three-Dimensional Coordinates Between Bodies of Arbitrary Shapes", MSSU-EIRS-80-7, October (1980).

2. "Tensors and Differential Geometry Applied to Analytic and Numerical Coordiate Generation", MSSU-EIRS-81-1, January (1981).

3. "A Non-Iterative Method for the Generation of Orthogonal Coordinate in Doubly-Connected Regions", Mathematics of Computation, Vol. 38, No. 158, (1982), pp. 501-516.

4. "Basic Differential Models for Coordinate Generation", Numerical Grid Generation, Edited by J. F. Thomspon, Elsevier Science Publishing Co., (1982), pp. 41-77.

5. "Numerical Generation of Three-Dimensional Coordinates Between Bodies of Arbitrary Shapes", Ibid, pp. 717-728.

6. "Boundary-Fitted Coordinate Systems for Numerical Solution of Partial Differential Equations - A Review", J. Computational Physics, Vol. 47, No. 1, (1982), pp. 1-108.

8. "Three-Dimensional Grid Generation from Elliptic systems", presented at the AIAA 6th Computational Fluid Dynamics Conference at Danver, MA, July 1983.

9. "Generation of Three-Dimensional Grids Through Elliptic Differential Equations", Von Karman Institute Lecture Series, March 1984.

10. "The Importance of Dynamically-Adaptive Grids In The Numerical Solution of partial Differential Equations", Von Karman Institute Lecture Series, March 1984.

11. "Numerical Generation of Orthogonal and Non-Orthogonal Coordinates In Two-Dimensional Simply- and Doubly-Connected Regions", Von Karman Institute Tech., Note No. 151, May 1984.

12. Computer Simulation of Three-Dimensional Grids", Society for Computer Simulation, Feb. (1984).

QUARTERLY OF APPLIED MATHEMATICS
JULY 1983

A NOTE ON THE MATHEMATICAL FORMULATION OF THE PROBLEM OF NUMERICAL COORDINATE GENERATION*

By

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Abstract. A set of second order partial differential equations for the generation of three-dimensional grids around and between arbitrary shaped bodies has been proposed. These equations basically depend on the Gauss equations for a surface and have been structured in such a way that an automatic connection is established between the succeeding generated surfaces.

The vanishing of the Riemann curvature tensor has been used to isolate those fundamental equations which every coordinate system in either two- or three-dimensional Euclidean space must satisfy.

1. Introduction. The problem of generating spatial coordinates by numerical methods is a problem of much interest in practically all branches of engineering science and physics. At present a number of techniques are under active development for the generation of two- and three-dimensional coordinates around and between bodies of arbitrary shapes. Among these efforts two easily discernable groups can be formed: (i) the methods based on the solution of certain PDE's, preferably of the elliptic type, and (ii) the algebraic methods. In a large number of methods in the first group a set of inhomogeneous Laplace equations is taken as the basic generating system. These equations are then inverted and solved for the Cartesian coordinates in the transformed plane. Based on this line of approach which started with the work of Winslow [1] some very practical results, particularly in two dimensions, have been obtained by Thompson et al. [2] and others.[1] For the generation of plane curvilinear coordinates some authors have also used hyperbolic and parabolic systems of equations, [3]. For the methods in the second group, i.e., the algebraic methods, refer to [3].

In this paper we have first considered the formulation of a $3D$ grid generation scheme which is basically dependent on the Gauss' equations for a surface. In this scheme a series

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[1] Refer to [3] for an extensive bibliography on the subject.

of surfaces are generated based on the given data of arbitrary shaped inner and outer surfaces. The method has also been structured in such a way that the variation of the third coordinate from one generated surface to the next is fully reflected in the system of generating equations. It has been found that these generating system of equations are simple to implement numerically. In particular, the solution of the proposed equations tends to the solution of the Laplace equations in the transformed plane in case the surface becomes a Cartesian plane. An exact solution of these equations for the case of three-dimensional curvilinear coordinates between a prolate ellipsoid and a sphere has been obtained.

In a plane, or a surface, or a $3D$ space there are endless possibilities of introducing either orthogonal or non-orthogonal coordinates. This realization imparts a sense of arbitrariness to the choice of the method to be used for coordinate generation. If it is *a priori* decided that the method should be based on solving partial differential equations then the arbitrariness in the selection of the appropriate equations for the generation of coordinates becomes a problem to be resolved. In Sec. 3 of this paper it has been shown that despite this arbitrariness it is possible to uncover certain sets of equations which must invariably be satisfied no matter which equations or methods have been used to generate the coordinates. For a detailed discussion of the methods discussed here and on some numerical results refer to Warsi [4–7].

2. Generating system based on the Gauss' equations. In the process of formulation of a $3D$ coordinate generation problem it is helpful to imagine the coordinates of a point in space as the intersection of three distinct surfaces on each of which one coordinate is held fixed. Using the convention of a right-handed curvilinear coordinate system x^1, x^2, x^3 or ξ, η, ζ (refer to Fig. 1) we introduce the coordinates in the surface $x^\nu = \text{const.}$ through the following scheme.

$$\nu = 1: (x^2, x^3); \qquad \nu = 2: (x^3, x^1); \qquad \nu = 3: (x^1, x^2).$$

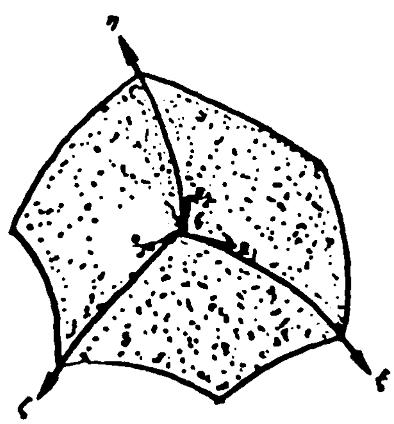

FIG. 1. Right handed coordinate arrangement and basis vectors

Thus the unit normal vector on the surface $x^\nu = \text{const.}$ is

$$\mathbf{n}^{(\nu)} = (\mathbf{r}_\alpha \times \mathbf{r}_\beta)/|\mathbf{r}_\alpha \times \mathbf{r}_\beta| \tag{1}$$

where

$$\nu = 1: \alpha = 2, \beta = 3; \qquad \nu = 2: \alpha = 3, \beta = 1; \qquad \nu = 3: \alpha = 1, \beta = 2. \tag{2}$$

From elementary differential geometry [8] we have the result that the rectangular Cartesian coordinates $\mathbf{r} = (x, y, z)$ or (x_1, x_2, x_3) of any point on every surface embedded in an Euclidean E_3 must satisfy the equations of Gauss. The Gauss equations for a surface $x^\nu = \text{const.}$ are given by

$$\mathbf{r}_{\alpha\beta} = \Gamma^\delta_{\alpha\beta}\mathbf{r}_\delta + b_{\alpha\beta}\mathbf{n}^{(\nu)}, \tag{3}$$

where all the Greek indices except ν can assume only two values. The values of α, β and the range of δ with ν follow the scheme given in (2). In Eq. (3),

$$\mathbf{r}_\delta = \frac{\partial \mathbf{r}}{\partial x^\delta}, \qquad \mathbf{r}_{\alpha\beta} = \frac{\partial^2 \mathbf{r}}{\partial x^\alpha \partial x^\beta},$$

$\Gamma^\delta_{\alpha\beta}$ are the surface Christoffel symbols of the second kind,[2] i.e.,

$$\Gamma^\delta_{\alpha\beta} = g^{\sigma\delta}[\alpha\beta, \sigma], \tag{4a}$$

$$[\alpha\beta, \sigma] = \frac{1}{2}\left(\frac{\partial g_{\alpha\sigma}}{\partial x^\beta} + \frac{\partial g_{\beta\sigma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right), \tag{4b}$$

and $b_{\alpha\beta}$ are the coefficients of the second fundamental form. Since on the surface $x^\nu = \text{const.}$ the vector $\mathbf{n}^{(\nu)}$ is orthogonal to the surface vectors $\mathbf{r}_\delta$,

$$b_{\alpha\beta} = \mathbf{n}^{(\nu)} \cdot \mathbf{r}_{\alpha\beta}. \tag{5}$$

For the purpose of a clear notation we denote the space Christoffel symbols as

$$\Gamma^l_{ij} = g^{kl}[ij, k], \tag{6a}$$

where

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right). \tag{6b}$$

Using (6a), we have

$$\mathbf{r}_{ij} = \frac{\partial^2 \mathbf{r}}{\partial x^i \partial x^j} = \Gamma^l_{ij}\mathbf{r}_l, \tag{7}$$

where all the Latin indices assume three values.

To fix ideas, we envisage a surface which is formed of the surface coordinates (ξ, η) and on which $\zeta = \text{const.}$ Dropping the index ν in Eq. (3), the three equations for the second derivatives of the Cartesian coordinates are

$$\mathbf{r}_{\xi\xi} = \Gamma^\delta_{11}\mathbf{r}_\delta + b_{11}\mathbf{n}, \quad \mathbf{r}_{\xi\eta} = \Gamma^\delta_{12}\mathbf{r}_\delta + b_{12}\mathbf{n}, \quad \mathbf{r}_{\eta\eta} = \Gamma^\delta_{22}\mathbf{r}_\delta + b_{22}\mathbf{n}. \tag{8a, b, c}$$

In Eqs. (8) $\mathbf{n}$ is orthogonal to both $\mathbf{r}_\xi$ and $\mathbf{r}_\eta$ and the coefficients of the first fundamental

[2] Refer to Appendix A for a collection of other formulae

form g_{11}, g_{12}, g_{22} are assumed to have been evaluated at $\zeta = \text{const}$. Obviously

$$g_{11} = x_\xi^2 + y_\xi^2 + z_\xi^2, \quad g_{12} = x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta, \quad g_{22} = x_\eta^2 + y_\eta^2 + z_\eta^2. \tag{9}$$

Multiplying Eq. (8a) by g_{22}, Eq. (8b) by $-2g_{12}$, Eq. (8c) by g_{11} and adding the three equations, we get

$$\mathfrak{L}\mathbf{r} + \left[(\Delta_2\xi)\mathbf{r}_\xi + (\Delta_2\eta)\mathbf{r}_\eta\right]G_3 = \mathbf{n}R, \tag{10}$$

where

$$\mathfrak{L} = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta}, \tag{11a}$$

$$\Delta_2 = \left[\partial_\xi\left\{\left(g_{22}\partial_\xi - g_{12}\partial_\eta\right)/\sqrt{G_3}\right\} + \partial_\eta\left\{\left(g_{11}\partial_\eta - g_{12}\partial_\xi\right)/\sqrt{G_3}\right\}\right]/\sqrt{G_3}, \tag{11b}$$

$$G_3 = g_{11}g_{22} - (g_{12})^2, \tag{11c}$$

$$\mathbf{n} = (X, Y, Z),$$

$$X = (y_\xi z_\eta - y_\eta z_\xi)/\sqrt{G_3}, \quad Y = (x_\eta z_\xi - x_\xi z_\eta)/\sqrt{G_3}, \quad Z = (x_\xi y_\eta - x_\eta y_\xi)/\sqrt{G_3}, \tag{11d}$$

$$R = (g_{22}b_{11} - 2g_{12}b_{12} + g_{11}b_{22}) = G_3(k_1 + k_2), \tag{11e}$$

where $k_1 + k_2$ is twice the mean curvature of the surface.

The operator $\Delta_2^{(3)}$ or simply Δ_2 defined in (11b) is the second order differential operator of Beltrami [8], for the surface $\zeta = \text{const}$. For any surface $x^\nu = \text{const}$. and following the scheme (2), we have

$$\Delta_2^{(\nu)} = \left[\partial_\alpha\left\{\left(g_{\beta\beta}\partial_\alpha - g_{\alpha\beta}\partial_\beta\right)/\sqrt{G_\nu}\right\} + \partial_\beta\left\{\left(g_{\alpha\alpha}\partial_\beta - g_{\alpha\beta}\partial_\alpha\right)/\sqrt{G_\nu}\right\}\right]/\sqrt{G_\nu}, \tag{12}$$

where G_ν are defined in Eq. (A.9). It is easy to show by using the surface Christoffel symbols $T_{\alpha\beta}^\delta$ that

$$\Delta_2\xi = \left(2g_{12}T_{12}^1 - g_{22}T_{11}^1 - g_{11}T_{22}^1\right)/G_3, \tag{13a}$$

$$\Delta_2\eta = \left(2g_{12}T_{12}^2 - g_{22}T_{11}^2 - g_{11}T_{22}^2\right)/G_3, \tag{13b}$$

where the metric coefficients $g_{\alpha\beta}$ are those as defined in (9). It is interesting to see that when the Laplacian operator ∇^2 for a two-dimensional Cartesian space is applied to the curvilinear coordinates (ξ, η) in an Euclidean plane, the resulting expressions are exactly of the same *form* as (13a, b), that is (refer to Eq. (A.13)),

$$\nabla^2\xi = \left(2g_{12}\Gamma_{12}^1 - g_{22}\Gamma_{11}^1 - g_{11}\Gamma_{22}^1\right)/(J)^2, \tag{14a}$$

$$\nabla^2\eta = \left(2g_{12}\Gamma_{12}^2 - g_{22}\Gamma_{11}^2 - g_{11}\Gamma_{22}^2\right)/(J)^2, \tag{14b}$$

where now

$$g_{11} = x_\xi^2 + y_\xi^2, \quad g_{12} = x_\xi x_\eta + y_\xi y_\eta, \quad g_{22} = x_\eta^2 + y_\eta^2, \quad J = x_\xi y_\eta - x_\eta y_\xi.$$

Though the right-hand side term R defined in (11e) has the necessary extrinsic effects, nevertheless we must have an explicit dependence of $\mathbf{r} = (x, y, z)$ on the third coordinate ζ. Thus using Eq. (A.11) we have

$$\mathbf{r}_{\xi\xi} = \Gamma^1_{11}\mathbf{r}_\xi + \Gamma^2_{11}\mathbf{r}_\eta + \Gamma^3_{11}\mathbf{r}_\zeta, \tag{15a}$$

$$\mathbf{r}_{\xi\eta} = \Gamma^1_{12}\mathbf{r}_\xi + \Gamma^2_{12}\mathbf{r}_\eta + \Gamma^3_{12}\mathbf{r}_\zeta, \tag{15b}$$

$$\mathbf{r}_{\eta\eta} = \Gamma^1_{22}\mathbf{r}_\xi + \Gamma^2_{22}\mathbf{r}_\eta + \Gamma^3_{22}\mathbf{r}_\zeta, \tag{15c}$$

and we evaluate each of these derivatives at ζ = const. Taking the dot product of Eqs. (15) with $\mathbf{n}$ and comparing with Eqs. (5), we get

$$b_{11} = \lambda\Gamma^3_{11}, \quad b_{12} = \lambda\Gamma^3_{12}, \quad b_{22} = \lambda\Gamma^3_{22}, \tag{16a}$$

where

$$\lambda = \mathbf{n}\cdot\mathbf{r}_\zeta = Xx_\zeta + Yy_\zeta + Zz_\zeta. \tag{16b}$$

Thus, the expression (11e) for R is replaced by

$$R = \lambda\left[g_{11}\Gamma^3_{22} - 2g_{12}\Gamma^3_{12} + g_{22}\Gamma^3_{11}\right]. \tag{17}$$

2.1 *Fundamental generating system of equations.* We now impose the following differential constraints on the coordinates ξ and η:

$$\Delta_2\xi = 0, \qquad \Delta_2\eta = 0, \tag{18}$$

and take them as the basic generating equations for the coordinates in a surface. A comparison of Eqs. (13) and (14) has already shown that Δ_2 is not a $2D$ Laplace operator except when the surface degenerates into a plane having no dependence on the z-coordinate.

It is a well-known result in differential geometry that the isothermic coordinates in a surface satisfy Eqs. (18) identically. The isothermic coordinates ξ and η are those orthogonal coordinates in a surface which yield $g_{22} = g_{11}$. The situation here is parallel to the choice of the Laplace equations $\nabla^2\xi = 0$, $\nabla^2\eta = 0$ for the generation of plane curvilinear coordinates [2], which are also satisfied by the conformal coordinates in a plane. The important point to note here is that the satisfaction of the Laplace equations is a necessary but not a sufficient condition for the existence of conformal coordinates. Similarly, the satisfaction of equations (18) is a necessary but not a sufficient condition for the existence of isothermic coordinates. It would, therefore, be more meaningful if we interpret Eqs. (18) as providing a set of differential constraints[3] on the metric coefficients g_{11}, g_{12}, and g_{22} defined in (9).

Having chosen Eqs. (18) as the generating system, the equations for the determination of the Cartesian coordinates, viz., Eq. (10) becomes

$$\mathcal{L}\mathbf{r} = \mathbf{n}R, \tag{19}$$

where $\mathcal{L}$, $\mathbf{n}$ and R have been defined in (11a), (11d), and (17) respectively. The three scalar equations in expanded form are now

$$g_{22}x_{\xi\xi} - 2g_{12}x_{\xi\eta} + g_{11}x_{\eta\eta} = XR, \tag{20a}$$

$$g_{22}y_{\xi\xi} - 2g_{12}y_{\xi\eta} + g_{11}y_{\eta\eta} = YR, \tag{20b}$$

[3] A manifestation of the many possibilities for introducing coordinates in a given place. space

$$g_{22}z_{\xi\xi} - 2g_{12}z_{\xi\eta} + g_{11}z_{\eta\eta} = ZR. \tag{20c}$$

For a plane $z = \text{const.}$, $R = 0$ and the Eqs. (20) are the inversions of the Laplace equations in the $\xi\eta$-plane.

It can be shown that Eqs. (20) can be combined to obtain the equations of a surface $z = z(x, y)$ in the well known form,

$$\alpha z_{xx} - 2\beta z_{xy} + \gamma z_{yy} = 2HM, \tag{21}$$

where

$$2H = k_1 + k_2 = R/G_3, \quad M = 1 + p^2 + q^2, \quad p = z_x, q = z_y,$$
$$\alpha = (1 + q^2)/\sqrt{M}, \quad \beta = pq/\sqrt{M}, \quad \gamma = (1 + p^2)/\sqrt{M}.$$

Using the following definitions and identities

$$G_3 = g_{11}g_{22} - (g_{12})^2, \quad X = -p/\sqrt{M}, \quad Y = -q/\sqrt{M}, \quad Z = 1/\sqrt{M},$$
$$\Delta_1(x, x) = (1 - X^2)G_3, \quad \Delta_1(x, y) = -XYG_3, \quad \Delta_1(y, y) = (1 - Y^2)G_3,$$

where

$$\Delta_1(a, b) = g_{22}a_\xi b_\xi - g_{12}(a_\xi b_\eta + a_\eta b_\xi) + g_{11}a_\eta b_\eta.$$

calculating $z_{\xi\xi}$, $z_{\xi\eta}$, $z_{\eta\eta}$ from z_ξ, z_η and substituting these expressions in (20c) while using Eqs.(20a, b), we get Eq. (21).

We now use the result that if (ξ, η) is a permissible system of coordinates in a surface then so is $(\bar{\xi}, \bar{\eta})$, where $\bar{\xi} = \bar{\xi}(\xi, \eta)$, $\bar{\eta} = \bar{\eta}(\xi, \eta)$, provided that the Jacobian of the transformation does not vanish. It is a straight forward matter to show that on coordinate transformation, Eqs. (20) become

$$\bar{\mathfrak{L}}x = \bar{X}\bar{R}, \quad \bar{\mathfrak{L}}y = \bar{Y}\bar{R}, \quad \bar{\mathfrak{L}}z = \bar{Z}\bar{R}, \tag{22a, b, c}$$

where

$$\bar{\mathfrak{L}} = \bar{g}_{22}\partial_{\bar{\xi}\bar{\xi}} - 2\bar{g}_{12}\partial_{\bar{\xi}\bar{\eta}} + \bar{g}_{11}\partial_{\bar{\eta}\bar{\eta}} + \bar{P}\partial_{\bar{\xi}} + \bar{Q}\partial_{\bar{\eta}}, \tag{23a}$$
$$\bar{P} = \bar{g}_{22}P^1_{11} - 2\bar{g}_{12}P^1_{12} + \bar{g}_{11}P^1_{22}, \tag{23b}$$
$$\bar{Q} = \bar{g}_{22}P^2_{11} - 2\bar{g}_{12}P^2_{12} + \bar{g}_{11}P^2_{22}, \tag{23c}$$
$$P^{\gamma}_{\mu\sigma} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial \bar{x}^{\sigma}}\frac{\partial^2 \bar{x}^{\gamma}}{\partial x^{\alpha}\partial x^{\beta}}, \tag{23d}$$

and $\bar{X}$, $\bar{Y}$, $\bar{Z}$, and $\bar{R}$ are exactly the same expressions as given in (11d) and (17), in the $\bar{x}^{\alpha}$ coordinate system. It is preferable to solve Eqs. (22) with $P^{\alpha}_{\beta\delta}$ as arbitrarily prescribed functions of the coordinates. This aspect of the formulation therefore provides a capability to redistribute the coordinate systems in the surface in any desired manner.[4]

2.2 *Example of a solution algorithm.* The discussion that follows pertains to the case when it is desired to generate the 3D curvilinear coordinates between two arbitrary shaped smooth surfaces. Let the surface coordinates (ζ, ξ) of the inner body $\eta = \eta_B$ and of the

[4] For a limiting form of Eqs (22) refer to Appendix B.

outer body $\eta = \eta_\infty$ be the same coordinates. Thus

$$\eta = \eta_B : \mathbf{r} = \mathbf{r}_B(\xi, \zeta); \qquad \eta = \eta_\infty : \mathbf{r} = \mathbf{r}_\infty(\xi, \zeta)$$

are known functions (either analytically or numerically) and hence the needed partial derivatives with respect to ξ and ζ are directly available at the surfaces.

For the computation of $\mathbf{r}_\zeta$ in the field one must first note that the coordinate ζ may not, in general, satisfy the equation $\Delta_2^{(2)}\zeta = 0$. Consequently, $\mathbf{r}_\zeta$ must satisfy the equation

$$\mathfrak{L}^{(2)}\mathbf{r} + G_2\left(\Delta_2^{(2)}\zeta\right)\mathbf{r}_\zeta = G_2\left(k_1^{(2)} + k_2^{(2)}\right)\mathbf{n}^{(2)}. \tag{24}$$

From this equation we devise a weighted integral formula[5]

$$\mathbf{r}_\zeta = \int\left[f_1(\eta)(\mathbf{r}_{\zeta\zeta})_B + f_2(\eta)(\mathbf{r}_{\zeta\zeta})_\infty\right]d\zeta, \tag{25a}$$

where

$$(\mathbf{r}_{\zeta\zeta})_{B,\infty} = \Bigg[\frac{G_2}{g_{11}}\left(k_1^{(2)} + k_2^{(2)}\right)\mathbf{n}^{(2)} + \frac{2g_{13}}{g_{11}}\mathbf{r}_{\xi\zeta} - \frac{g_{33}}{g_{11}}\mathbf{r}_{\xi\xi}$$
$$- \frac{\sqrt{G_2}}{g_{11}}\left\{\frac{\partial}{\partial\zeta}\left(\frac{g_{11}}{\sqrt{G_2}}\right) - \frac{\partial}{\partial\xi}\left(\frac{g_{13}}{\sqrt{G_2}}\right)\right\}\mathbf{r}_\zeta\Bigg]_{B,\infty}, \tag{25b}$$

and

$$f_1(\eta_B) = 1, \quad f_1(\eta_\infty) = 0, \quad f_2(\eta_B) = 0, \quad f_2(\eta_\infty) = 1.$$

There is no difficulty in the numerical evaluation of (25a) in an iterative cycle after the weighting functions f_1 and f_2 have been prescribed *a priori*.

Referring to Fig. (2a), we now solve Eqs. (20) or (22) for each ζ = const., by prescribing the values of x, y, z on the curves C_1 and C_2 which respectively represent the curves on B and ∞. In Fig. (2b) C_3 and C_4 are the cut lines on which periodic conditions are to be imposed.

2.3 *An exact solution of the proposed equations.* The following example demonstrates that the proposed set of generating equations (22) are consistent and provide nontrivial solutions.

We consider the case of coordinate generation between an inner body $\eta = \eta_B$ which is a prolate ellipsoid and an outer boundary $\eta = \eta_\infty$ which is a sphere. The coordinates which vary on these two surfaces are ξ and ζ. A curve C_1 on the inner surface for ζ = const. is

$$x = \tau\cosh\eta_B\cos\zeta, \quad y = \tau\sinh\eta_B\sin\zeta\cos\xi, \quad z = \tau\sinh\eta_B\sin\zeta\sin\xi, \tag{26a}$$

where τ and η_B are the parameters of the ellipsoid. Similarly the curve C_2 corresponding to the same ζ = const. on the outer surface is

$$x = \exp(\eta_\infty)\cos\zeta, \quad y = \exp(\eta_\infty)\sin\zeta\cos\xi, \quad z = \exp(\eta_\infty)\sin\zeta\sin\xi. \tag{26b}$$

[5] The discussion given here is directed to the situation of Fig. (2a). For other situations, e.g., simply-connected domains or multibody problems the method of calculating $\mathbf{r}_\zeta$ must always be devised separately. Note also that Eq. (24) reflects the condition $\Delta_2^{(2)}\xi = 0$.

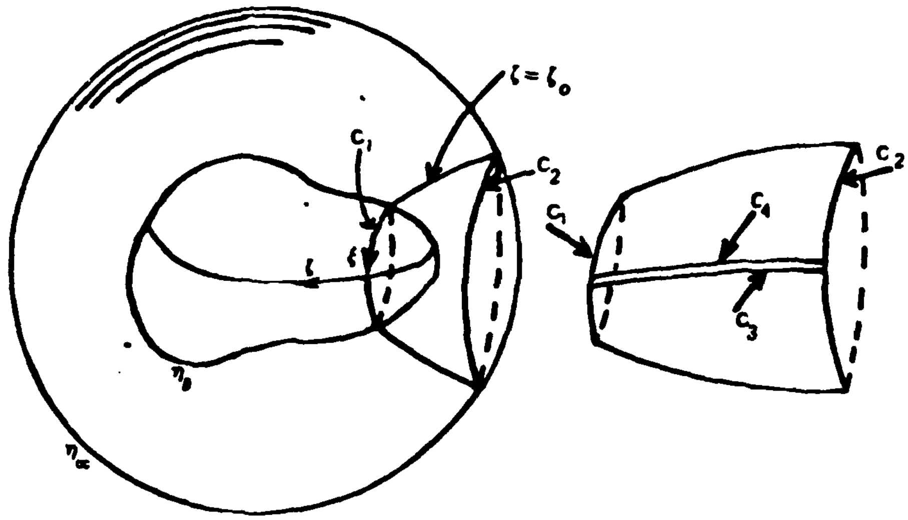

FIG. 2. (a) Topology of the given surfaces.
(b) Surfaces to be generated.

In order to generate surfaces on which ξ and η are the coordinates and in which the η-coordinate can nonuniformly be distributed (contraction or expansion in the η coordinates), we assume

$$\xi = \xi(\bar{\xi}), \qquad \eta = \eta(\bar{\eta}) + \eta_B, \tag{27}$$

where $\bar{\xi} = 0$ corresponding to $\xi = 0$ and $\bar{\eta} = \bar{\eta}_B$ corresponding to $\eta = \eta_B$. Thus $\xi(0) = 0$, $\eta(\bar{\eta}_B) = 0$. Under the transformation (27), the only nonzero components of $P^{\gamma}_{\mu\sigma}$ are P^1_{11} and P^2_{22}. Writing

$$\lambda(\bar{\xi}) = \frac{d\xi}{d\bar{\xi}}, \qquad \theta(\bar{\eta}) = \frac{d\eta}{d\bar{\eta}},$$

we have

$$P^1_{11} = -\frac{1}{\lambda}\frac{d\lambda}{d\bar{\xi}}, \qquad P^2_{22} = -\frac{1}{\theta}\frac{d\theta}{d\bar{\eta}}. \tag{28}$$

Based on the forms of the boundary conditions (26a, b) we assume the following forms for x, y, z for each $\zeta = \text{const.}$:

$$x = f(\bar{\eta})\cos\zeta, \quad y = \phi(\bar{\eta})\sin\zeta\cos\xi, \quad z = \phi(\bar{\eta})\sin\zeta\sin\xi. \tag{29}$$

The boundary conditions are

$$\begin{aligned} f(\bar{\eta}_B) &= \tau\cosh\eta_B, & f(\bar{\eta}_\infty) &= \exp(\eta_\infty), \\ \phi(\bar{\eta}_B) &= \tau\sinh\eta_B, & \phi(\bar{\eta}_\infty) &= \exp(\eta_\infty). \end{aligned} \tag{30}$$

Using the expressions of (29) we calculate the various partial derivatives, metric coefficients, and all other data as needed for Eqs. (22). On substitution we get an equation containing $\sin^2\zeta$ and $\cos^2\zeta$. Equating to zero the coefficients of $\sin^2\zeta$ and $\cos^2\zeta$, we obtain

$$f''/f' = \theta'/\theta + \phi'/\phi, \tag{31}$$

$$\phi''/\phi' = \theta'/\theta + \phi'/\phi, \tag{32}$$

where a prime denotes differentiation with respect to $\bar{\eta}$. On direct integrations of Eqs. (31) and (32) under the boundary conditions (30), we get

$$f(\bar{\eta}) = A\exp(B\eta(\bar{\eta})) + C, \tag{33a}$$

$$\phi(\bar{\eta}) = D\exp(B\eta(\bar{\eta})), \tag{33b}$$

where

$$A = \tau\big[(\exp(\eta_\infty) - \tau\cosh\eta_B)\sinh\eta_B\big]/(\exp(\eta_\infty) - \tau\sinh\eta_B),$$
$$B = (\eta_\infty - \ln(\tau\sinh\eta_B))/(\eta_\infty - \eta_B),$$
$$C = \tau\big[\exp(\eta_\infty)(\cosh\eta_B - \sinh\eta_B)\big]/(\exp(\eta_\infty) - \tau\sinh\eta_B),$$
$$D = \tau\sinh\eta_B.$$

As an application, we take

$$\xi(\bar{\xi}) = a\bar{\xi}, \qquad \eta(\bar{\eta}) = b(\bar{\eta} - \bar{\eta}_B)k^{\bar{\eta}}, \tag{34}$$

where a, b, and k are constants. Thus

$$\eta(\bar{\eta}) = \frac{(\eta_\infty - \eta_B)(\bar{\eta} - \bar{\eta}_B)}{\bar{\eta}_\infty - \bar{\eta}_B}k^{(\bar{\eta} - \bar{\eta}_\infty)}. \tag{35}$$

By taking a value of k slightly greater than one ($k = 1.05$) we can have sufficient contraction in the $\bar{\eta}$-coordinate near the inner surface. For the chosen problem since the dependence on ζ is simple, we find that the generated coordinates between a prolate ellipsoid and a sphere are

$$x = [A\exp(B\eta(\bar{\eta})) + C]\cos\zeta, \qquad y = D\exp(B\eta(\bar{\eta}))\sin\zeta\cos\xi,$$
$$z = D\exp(B\eta(\bar{\eta}))\sin\zeta\sin\xi. \tag{36}$$

This example also shows that the chosen generating system of equations (20) or (22) are capable of providing non-isothermic coordinates between smooth surfaces.

3. *Differential* equations based on the Riemann tensor. In any given space there are endless possibilities for the introduction of coordinate curves. Each chosen set of curves determines its own metric components. For example, in a Cartesian plane besides introducing rectangular Cartesian coordinates x, y, we also have endless possibilities for introducing either orthogonal or nonorthogonal coordinate curves. However, as is well known, there is a basic differential constraint on the variations of g_{ij}'s irrespective of the coordinate system. Since the curvature of an Euclidean two-dimensional plane is identically zero, the basic differential constraint on the g_{ij}'s is

$$(G_3)^{-1/2}R_{1212} = \frac{\partial}{\partial\eta}\left(\frac{\sqrt{G_3}}{g_{11}}\Gamma^2_{11}\right) - \frac{\partial}{\partial\xi}\left(\frac{\sqrt{G_3}}{g_{11}}\Gamma^2_{12}\right) = 0, \tag{37}$$

where ξ, η are any arbitrary coordinate curves in the plane. Thus no matter which coordinate system is introduced in a plane, the corresponding matrics g_{ij} must satisfy Eq.

(37). Equation (37) has also been used as the basic generating equation for the generation of orthogonal coordinates in a plane [9]. In general, the Riemann curvature tensor R_{rjnp} defined as,

$$R_{rjnp} = \frac{1}{2}\left(\frac{\partial^2 g_{rp}}{\partial x^j \partial x^n} + \frac{\partial^2 g_{jn}}{\partial x^r \partial x^p} - \frac{\partial^2 g_{rn}}{\partial x^j \partial x^p} - \frac{\partial^2 g_{jp}}{\partial x^r \partial x^n}\right) + g^{st}([jn, s][rp, t] - [jp, s][rn, t]) \tag{38}$$

defines the components of the curvature tensor of any general space. If the space is N-dimensional, then the number of components R_{rjnp} are given by $N^2(N^2 - 1)/12$. Thus for $N = 2$ there is one distinct surviving component stated in Eq. (37). However, for $N = 3$, it has six distinct components R_{1212}, R_{1313}, R_{2323}, R_{1213}, R_{1232}, R_{1323}. If the $3D$-space is Euclidean, then its curvature is zero, so that the six equations

$$\begin{aligned} R_{1212} &= 0, \quad R_{1313} = 0, \quad R_{2323} = 0, \\ R_{1213} &= 0, \quad R_{1232} = 0, \quad R_{1323} = 0 \end{aligned} \tag{39}$$

determine the differential constraints for the six metric coefficients g_{ij} in any coordinate system introduced in an Euclidean space. These equations in the expanded form have been given in [5] and [6].

Equations (39) are those consistent set of partial differential equations which must always be satisfied by the metric coefficients g_{ij}. In the $3D$ case Eqs. (39) are six equations in six unknowns, and, therefore, they form a closed system of equations. In contrast, for the $2D$ case there is only one equation (Eq. (37)) and three unknowns g_{11}, g_{12}, g_{22} and therefore some constraints are needed to turn Eq. (37) (such as orthogonality [9]) into a solvable equation.

To obtain the Cartesian coordinates on the basis of the available g_{ij}'s, we introduce the unit base vectors $\boldsymbol{\lambda}_i$ as

$$\boldsymbol{\lambda}_i = \mathbf{a}_i/\sqrt{g_{ii}}, \quad \text{no sum on } i. \tag{40}$$

Let the components of $\boldsymbol{\lambda}_i$ along the rectangular Cartesian axes be denoted as u_i, v_i, w_i, so that

$$\boldsymbol{\lambda}_i = (u_i, v_i, w_i),$$

where

$$\begin{aligned} u_1 &= x_\xi/\sqrt{g_{11}}, \quad v_1 = y_\xi/\sqrt{g_{11}}, \quad w_1 = z_\xi/\sqrt{g_{11}}, \\ u_2 &= x_\eta/\sqrt{g_{22}}, \quad v_2 = y_\eta/\sqrt{g_{22}}, \quad w_2 = z_\eta/\sqrt{g_{22}}, \\ u_3 &= x_\zeta/\sqrt{g_{33}}, \quad v_3 = y_\zeta/\sqrt{g_{22}}, \quad w_3 = z_\zeta/\sqrt{g_{33}}. \end{aligned} \tag{41}$$

If the components u_i, v_i, w_i become known by some method then it is possible to evaluate the Cartesian coordinates through the line integrals

$$\mathbf{r} = \int\left(\boldsymbol{\lambda}_1\sqrt{g_{11}}\,d\xi + \boldsymbol{\lambda}_2\sqrt{g_{22}}\,d\eta + \boldsymbol{\lambda}_3\sqrt{g_{33}}\,d\zeta\right). \tag{42}$$

The determination of u_i, v_i, w_i is a separate problem which we now consider. First of all using (40) in Eq. (A.11), we get a system of first order partial differential equations

$$\frac{\partial \lambda_i}{\partial x^j} = \lambda_1\left(\frac{g_{11}}{g_{ii}}\right)^{1/2}\Gamma^1_{ij} + \lambda_2\left(\frac{g_{22}}{g_{ii}}\right)^{1/2}\Gamma^2_{ij}$$

$$+\lambda_3\left(\frac{g_{33}}{g_{ii}}\right)^{1/2}\Gamma^3_{ij} - \frac{\lambda_i}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^j}, \tag{43}$$

where there is *no sum* on the repeated index i. Equations (43) form a system of 27 first order PDE's in nine independent variables u_i, v_i, w_i. This system of equations is *overdetermined* and thus its solvability should depend on certain compatibility conditions. According to a theorem on the overdetermined system of equations [10], if the compatibility conditions hold then the solution of Eqs. (43) exists and is unique. The conditions

$$\partial^2\lambda_i/\partial x^m \partial x^j = \partial^2\lambda_i/\partial x^j \partial x^m \tag{44}$$

for all values of i, m, and j are the compatibility conditions. To prove (44) we use Eq. (A.11), which on cross differentiation yields

$$\frac{\partial^2 \mathbf{a}_i}{\partial x^m \partial x^j} - \frac{\partial^2 \mathbf{a}_i}{\partial x^j \partial x^m} = R^l_{\cdot imj}\mathbf{a}_l, \tag{45}$$

where $R^l_{\cdot imj}$ is the Riemann-Christoffel curvature tensor and is related to the Riemann's tensor R_{ijkl}. Evidently in our present case $R^l_{\cdot imj} = 0$, since the space is Euclidean. Inserting (40) in (45) we find that Eq. (44) are *identically satisfied*.

It is interesting to note that for a two-dimensional curvilinear coordinate system there is no need to solve the system of equations such as (43). In this case the single differential equation with $G_3 = g$

$$R_{1212} = \sqrt{g}\left[\frac{\partial}{\partial \eta}\left(\frac{\sqrt{g}\,\Gamma^2_{11}}{g_{11}}\right) - \frac{\partial}{\partial \xi}\left(\frac{\sqrt{g}\,\Gamma^2_{12}}{g_{11}}\right)\right] = 0$$

implies the existence of a single function $\alpha(\xi, \eta)$ such that

$$\alpha_\xi = \frac{-\sqrt{g}}{g_{11}}\Gamma^2_{11}, \qquad \alpha_\eta = \frac{-\sqrt{g}}{g_{11}}\Gamma^2_{12}.$$

Consequently

$$u_1 = \cos\alpha, \quad v_1 = -\sin\alpha, \quad u_2 = \cos(\alpha - \theta), \quad v_2 = -\sin(\alpha - \theta),$$

where α is the angle made by the tangent to the coordinate line η = const. in a clockwise sense with the x-axis, and

$$\cos\theta = g_{12}/\sqrt{g_{11}g_{22}}$$

is known. The angle α becomes known since g_{ij} are known; e.g. [9].

3.1 *Case of orthogonal coordinates*. For orthogonal coordinates since the cosines of the angles between the coordinate curves are zero, we have

$$g_{12} = g_{13} = g_{23} = 0. \tag{46}$$

Consequently,

$$[12,3] = [13,2] = [23,1] = 0, \quad \Gamma^3_{12} = \Gamma^2_{13} = \Gamma^1_{23} = 0, \quad g = g_{11}g_{22}g_{33}.$$

The equations for the metric coefficients, viz. Eqs. (39) under the constraints of orthogonality (46) simply reduce to the Lame's equations. They can concisely be written as six equations by dropping the summation convention in the form

$$\frac{\partial}{\partial x^j}\left(\frac{1}{h_j}\frac{\partial h_k}{\partial x^j}\right) + \frac{\partial}{\partial x^k}\left(\frac{1}{h_k}\frac{\partial h_j}{\partial x^k}\right) + \frac{1}{h_i^2}\frac{\partial h_j}{\partial x^i}\frac{\partial h_k}{\partial x^i} = 0, \tag{47a}$$

$$\frac{\partial^2 h_i}{\partial x^j \partial x^k} = \frac{1}{h_j}\frac{\partial h_i}{\partial x^j}\frac{\partial h_j}{\partial x^k} + \frac{1}{h_k}\frac{\partial h_i}{\partial x^k}\frac{\partial h_k}{\partial x^j}, \tag{47b}$$

where (i, j, k) are to be taken in the cyclic permutations of $(1, 2, 3)$, in this order, and

$$h_1 = \sqrt{g_{11}}, \quad h_2 = \sqrt{g_{22}}, \quad h_3 = \sqrt{g_{33}}.$$

To obtain the differential equations for the Cartesian coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$, we first proceed from Eq. (A.13) and have

$$\sqrt{g}\,\nabla^2\xi = \frac{\partial}{\partial\xi}(h_2h_3/h_1), \quad \sqrt{g}\,\nabla^2\eta = \frac{\partial}{\partial\eta}(h_1h_3/h_2), \quad \sqrt{g}\,\nabla^2\zeta = \frac{\partial}{\partial\zeta}(h_1h_2/h_3), \tag{48}$$

where

$$\sqrt{g} = h_1h_2h_3, \qquad \nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz}.$$

Proceeding directly from Eq. (A.14) and using Eqs. (46) and (48), the equations for the Cartesian coordinates are

$$\Xi x_m = 0, \qquad m = 1,2,3, \tag{49}$$

where

$$\Xi = \frac{\partial}{\partial\xi}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial\zeta}\right).$$

Note that the operator Ξ and the Laplacian ∇^2 are related as

$$\Xi\phi = h_1h_2h_3\nabla^2\phi,$$

where ϕ is a scalar.

Equations (47) and (49) are those consistent set of equations which every orthogonal coordinate system must satisfy.

3.2 *The case of isothermic coordinates.* Isothermic coordinates in a surface embedded in a $3D$ Euclidean space are those coordinates in which the metric coefficients g_{11} and g_{33} in the surface η = const. are equal. That is, the element of length ds on η = const. is given by

$$(ds)^2_{\eta=\mathrm{const}} = g_{11}\left[(d\xi)^2 + (d\zeta)^2\right],$$

where ξ, ζ are chosen to be the surface coordinates. Using (46) and setting

$$g_{33} = g_{11} \quad \text{and} \quad g_{22} = F(\eta)$$

in Eqs. (39), we obtain the basic equations for g_{11}, which are

$$\frac{\partial}{\partial\xi}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\xi}\right)+\frac{\partial}{\partial\zeta}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\zeta}\right)+\frac{1}{2Fg_{11}}\left(\frac{\partial g_{11}}{\partial\eta}\right)^2=0, \tag{50a}$$

$$\frac{\partial}{\partial\eta}\left(\frac{1}{\sqrt{Fg_{11}}}\frac{\partial g_{11}}{\partial\eta}\right)=0, \tag{50b}$$

$$\frac{\partial}{\partial\zeta}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\eta}\right)=0, \tag{50c}$$

$$\frac{\partial}{\partial\xi}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\eta}\right)=0. \tag{50d}$$

It can easily be verified that the only solution of Eqs. (50c, d) is

$$g_{11}=[a+P(\eta)]^2 f(\xi,\zeta), \qquad a=\text{const}. \tag{51}$$

Thus from (50b)

$$F(\eta)=(dP/d\eta)^2. \tag{52}$$

Substituting (51) and (52) in Eq. (50a), the differential equation for $f(\xi,\zeta)$ becomes

$$\frac{\partial}{\partial\xi}\left(\frac{1}{f}\frac{\partial f}{\partial\xi}\right)+\frac{\partial}{\partial\zeta}\left(\frac{1}{f}\frac{\partial f}{\partial\zeta}\right)+2f=0. \tag{53}$$

In Kreyszig [11], we have the result that if in a portion of a surface isothermic coordinates can be introduced then that portion of the surface can conformally be mapped onto a plane. Thus in effect the solution of Eq. (53) provides that mapping function which conformally maps a surface onto a plane. As a verification of the above conclusion, we verify that the function

$$f=4e^{2\zeta}/(1+e^{2\zeta})^2 \tag{54}$$

is a solution of Eq. (53). This function is related with the isothermic coordinates on a sphere. Using the parametric equations of a sphere

$$x=[a+P(\eta)]\cos\theta,\quad y=[a+P(\eta)]\sin\theta\sin\phi,\quad z=[a+P(\eta)]\sin\theta\cos\phi$$

and writing

$$\xi=\phi,\qquad \zeta=\ln\tan\frac{\theta}{2},$$

where $0<\phi<2\pi$ and $0<\theta<\pi$, we obtain

$$g_{33}=g_{11}=4(a+P)^2e^{2\zeta}/(1+e^{2\zeta})^2.$$

Thus the equations

$$\begin{aligned} x&=(a+P)(1-e^{2\zeta})/(1+e^{2\zeta}),\\ y&=2(a+P)e^{\zeta}\sin\xi/(1+e^{2\zeta}),\\ z&=2(a+P)e^{\zeta}\cos\xi/(1+e^{2\zeta}) \end{aligned} \tag{55}$$

represent a sphere of radius $a + P(\eta)$ in terms of the isothermic coordinates ξ, ζ in the surface. Since $P(\eta)$ is an arbitrary function of η, we now have the capability of prescribing a suitable function $P(\eta)$ to achieve any sort of contraction on expansion in the field. It is expected that the representation (55) should prove useful in the computational problems associated with a sphere.

4. Conclusions. In Sec. 2 of this paper a set of second order PDE's have been obtained which generate a series of surfaces between the given inner and outer arbitrary shaped bodies. The necessary mathematical apparatus which connects one generated surface with its neighbor along with the curvature properties of each surface has been incorporated in the right hand side terms of the equations. (Eqs. (20) or (22)). By changing the computational techniques these equations can also be used to generate the $3D$ coordinates when more than one inner bodies are present in the field.

In Sec. 3, based on some basic differential geometric concepts, a number of field equations have been isolated which must always be satisfied by any coordinate system in an Euclidean space. Efficient numerical methods are to be developed to solve these quasilinear equations (Eqs. (39)) on a digital computer.

Appendix A. In this appendix we collect some useful formulae which have been used in the main text.

As noted in the text, a general curvilinear coordinate system is denoted as x^i, $i = 1, 2, 3$, or as ξ, η, ζ, while a rectangular Cartesian system is denoted as x_m, $m = 1, 2, 3$ or as x, y, z. Since $\mathbf{r}$ is a position vector in an Euclidean space, the covariant base vectors $\mathbf{a}_i$ are given by

$$\mathbf{a}_i = \partial \mathbf{r}/\partial x^i, \tag{A.1}$$

while the contravariant base vectors $\mathbf{a}^i$ are given by

$$\mathbf{a}^i = \operatorname{grad} x^i. \tag{A.2}$$

The covariant and the contravariant metric components are respectively given by

$$g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j, \qquad g^{ij} = \mathbf{a}^i \cdot \mathbf{a}^j. \tag{A.3}$$

Both metric coefficients are related through the equations

$$g^{ij} g_{ik} = \delta^j_k, \tag{A.4}$$

where δ^j_k are the Kronecker deltas. Also

$$g = \det(g^{ij}), \qquad g\hat{g} = 1. \tag{A.5}$$

Based on (A.4), we also have

$$\mathbf{a}^i = g^{ij}\mathbf{a}_j \tag{A.6}$$

$$= e^{ijk}(\mathbf{a}_j \times \mathbf{a}_k)/2\sqrt{g}, \tag{A.7}$$

where, here and in all the expressions a repeated lower and upper index always stands for a sum over the range of index values. Also

$$g = \det(g_{ij}) = g_{33}G_3 + g_{13}G_5 + g_{23}G_6$$
$$= g_{22}G_2 + g_{12}G_4 + g_{23}G_6 = g_{11}G_1 + g_{12}G_4 + g_{13}G_5, \tag{A.8}$$

where

$$G_1 = g_{22}g_{33} - (g_{23})^2, \quad G_2 = g_{11}g_{33} - (g_{13})^2, \quad G_3 = g_{11}g_{22} - (g_{12})^2,$$
$$G_4 = g_{13}g_{23} - g_{12}g_{33}, \quad G_5 = g_{12}g_{23} - g_{13}g_{22}, \quad G_6 = g_{12}g_{13} - g_{23}g_{11}. \tag{A.9}$$

Note that

$$g^{11} = G_1/g, \quad g^{22} = G_2/g, \quad g^{33} = G_3/g,$$
$$g^{12} = G_4/g, \quad g^{13} = G_5/g, \quad g^{23} = G_6/g. \tag{A.10}$$

The derivative of a covariant base vector is given by

$$\partial \mathbf{a}_i/\partial x^j = \partial^2 \mathbf{r}/\partial x^i \partial x^j = \Gamma^l_{ij}\mathbf{a}_l. \tag{A.11}$$

The Laplacian of a scalar ϕ in a curvilinear coordinate system is

$$\nabla^2\phi = g^{ij}\left(\frac{\partial^2\phi}{\partial x^i \partial x^j} - \Gamma^r_{ij}\frac{\partial\phi}{\partial x^r}\right), \tag{A.12}$$

where Γ^r_{ij} have been defined in Eqs. (6).

If $\phi = x^m$ is any curvilinear coordinate then from (A.12)

$$\nabla^2 x^m = -g^{ij}\Gamma^m_{ij}. \tag{A.13}$$

If $\phi = x_m$ is any rectangular Cartesian coordinate then from (A.12)

$$g^{ij}\frac{\partial^2 x_m}{\partial x^i \partial x^j} + (\nabla^2 x^r)\frac{\partial x_m}{\partial x^r} = 0. \tag{A.14}$$

Appendix B. In numerical computations it is desirable to solve Eqs. (22) in their limiting forms on certain special lines in the field. Referring to Fig. (2a), let the x-axis be aligned to pass through the inner body from two of its points, which, when extended in both directions meets the outer body at its two corresponding points. The portions of the lines between the inner and the outer bodies form the right and the left segments. On each segment $y = z = 0$, and according to the adopted convention $\zeta = 0$ and $\zeta = \pi$ on the right and the left segments respectively for all values of ξ. With this choice of the axes only Eq. (22a) is of interest. Taking the limit of Eq. (22a) as $\zeta \to 0$ or $\zeta \to \pi$, we obtain

$$x_{\eta\eta} + P^2_{22}x_\eta = \lim_{\substack{\zeta \to 0 \\ \text{or} \\ \zeta \to \pi}} \left(\bar{g}\bar{\lambda}\,\bar{g}_{22}\bar{\Gamma}^3_{11}/\bar{g}_{11}\right). \tag{B.1}$$

where the control function P^2_{22} has already been chosen *a priori*. The terms on the right hand side of Eq. (B.1) are difficult to assess for their limiting behaviors. However, some guidance can be obtained from the exact solution discussed in Sec. 2.3. This approach suggests that in any case, the following estimates can be used.

$$\bar{X} = f_1(\bar{\eta})(\bar{g}_{11})^{1/2}/x_{\bar{\eta}}, \quad \bar{\lambda} = f_2(\bar{\eta}),$$

$$\bar{\Gamma}^3_{11} = f_3(\bar{\eta})(\bar{g}_{11})^{1/2}, \quad \bar{g}_{22} = x^2_{\bar{\eta}}, \qquad \text{for } \zeta \to 0 \text{ or } \pi, \quad \text{(B.2)}$$

where f_1, f_2, f_3 are functions of $\bar{\eta}$. Using the estimates (B.2) in Eq. (B.1), we obtain

$$x_{\bar{\eta}\bar{\eta}} + T(\bar{\eta})x_{\bar{\eta}} = 0, \qquad \text{(B.3)}$$

where

$$T(\bar{\eta}) = P^2_{22} - F(\bar{\eta}), \qquad F(\bar{\eta}) = f_1 f_2 f_3.$$

The scheme now is to solve Eq. (B.3) by prescribing $T(\bar{\eta}) \neq P^2_{22}$ arbitrarily to achieve the desired control of points on the segments. Since P^2_{22} has already been chosen in advance this approach produces those values of $F(\bar{\eta})$ (though they need not be calculated) which are consistent with the basic equation, viz., Eq. (22a).

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APPENDIX B

Numerical Grid Generation in Arbitrary Surfaces Through A Second Order Differential-Geometric Model

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ABSTRACT

In this paper a set of second-order partial differential equations for the generation of coordinates in a given surface have been developed and then solved numerically to demonstrate their versatality. The proposed equations are not some arbitrarily chosen equations but are a consequence of the formulae of Gauss and thus carry with them an explicit dependence on the geometric properties of the given surface. Furthermore, these equations are easy to solve and require only the specification of the bounding curves to provide the Dirichlet boundary conditions for their solution. Results of coordinate generation both in the simply and doubly-connected regions on some known surfaces, with the option of coordinate redistribution, have been presented. Extension of this technique to arbitrary surfaces seems to be straightforward.

I. INTRODUCTION

The problem of generating spatial coordinates by numerical methods through carefully selected mathematical models is of current interest both in mechanics and physics. A review of various methods of coordinate generation in both two and three-dimensional Euclidean spaces is available in [1], and reference may also be made to the proceedings of two recent conferences, [2], [3], on the topic of numerical grid generation.

This paper is exclusively directed to the problem of generation of a desired coordinate system in the surface of a given body and thus, in a basic sense, it is an effort directed at the problem of grid generation in a two-dimensional non-Euclidean space. The mathematical model selected for this purpose is based on the formulae of Gauss for a surface and has been discussed by the author in earlier publications, [4] - [7]. The resulting equations are three coupled quasilinear elliptic partial differential equations with the Cartesian coordinates as the independent variables. These equations are nonhomogeneous with the righthand sides depending both on the components of the normal and the mean curvature of the surface. These equations therefore reflect the geometrical aspect of the surface in an explicit manner.

The proposed equations have been used to generate three-dimensional coordinates between two given surfaces by using an extrinsic form of the mean curvature, [7], [8], [9]. However, if the purpose is to generate the coordinates only in a given surface then the intrinsic form of the mean curvature has to be used, as has been done in this paper.

Previous work on the subject of grid generation in surfaces has been done by using either the algebraic techniques, [10] - [12] or using the PDE approach, [13] - [16]*. All these methods depend very heavily on the use of highly accurate interpolating schemes. On the other hand, the method proposed here depends only on the availability of the surface equation in the cartesian form and on the prescription of the data on the bounding curves in the surface which eventually form the Dirichlet boundary conditions for the proposed equations.

Numerical solutions of the proposed equations for the coordinates in either simply or doubly-connected regions of some known surfaces have been obtained and shown in Figs. (1) - (12). It has also been shown that any desired control on the distribution of grid spacing can be excerised by a proper choice of the control functions, cf. Figs. (7) - (9). Extension of the proposed method to arbitrary surfaces is purely formal.

* It has been shown in [16] that the equations proposed in [13] can be directly obtained by using equations (4.10) - (4.12) of sect. IV.

II. NOMENCLATURE

$b_{\alpha\beta} = \underset{\sim}{n}^{(\nu)} \cdot \underline{r}_{,\alpha\beta}$: coefficients of the second fundamental form in the surface ν = const.

D second-order differential operator; (Eq. 3.3)

$g = \det(g_{ij})$

$G_\nu = g_{\alpha\alpha}g_{\beta\beta} - (g_{\alpha\beta})^2$; ν = 1, 2, 3.

g_{ij} or $g_{\alpha\beta}$ covariant metric coefficients

g^{ij} or $g^{\alpha\beta}$ Contravariant metric coefficients

$J_\nu = \sqrt{G_\nu}$ Surface Jacobian

$k_I^{(\nu)}$, $k_{II}^{(\nu)}$ Principal curvatures at a point in the surface ν = const.

L Second-order differential operator; (Eq. 4.13)

$\underline{n}^{(\nu)}$ Unit normal vector on the surface ν = const.

$\overline{P}$, $\overline{Q}$, $P^{\alpha}_{\beta\gamma}$ Control functions

x^i 3D Curvilinear Coordinates

x^α 2D Curvilinear Coordinates

x_i 3D Rectangular Cartesian coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$

$X_i^{(\nu)}$ Rectangular components of $\underline{n}^{(\nu)}$; $X_1^{(\nu)} = X^{(\nu)}$, $X_2^{(\nu)} = Y^{(\nu)}$, $X_3^{(\nu)} = Z^{(\nu)}$.

$$T^{\delta}_{\alpha\beta} = \frac{1}{2} g^{\sigma\delta} \left(\frac{\partial g_{\alpha\sigma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\sigma}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\sigma}}\right),$$ the surface Christoffel symbols of the second kind.

$$\Gamma^{\ell}_{ij} = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{ik}}{\partial x^{j}} + \frac{\partial g_{jk}}{\partial x^{i}} - \frac{g_{ij}}{\partial x^{k}}\right),$$ the space christoffel of the second kind.

$$\Delta_2^{(\nu)} x^{\alpha} = -g^{\beta\gamma} T^{\alpha}_{\beta\gamma},$$ Beltrami's second-order differential parameter

(i) Notation For Partial Derivatives:

$$\underset{\sim}{r}_{,\alpha} = \frac{\partial \underset{\sim}{r}}{\partial x^{\alpha}}; \quad \underset{\sim}{r}_{,\alpha\beta} = \frac{\partial^2 \underset{\sim}{r}}{\partial x^{\alpha} \partial x^{\beta}}; \quad x_{p,\alpha} = \frac{\partial x_p}{\partial x^{\alpha}} \quad \text{etc.}$$

(ii) Note On The Use of Indices:

The Latin indices i, j, k etc. are used when the index range is from 1 to 3. The Greek indices α, β, γ, etc. (except ν, see below) are used for the cases when the indices assume only two integer values.

$\nu = 1$: α,β etc. assume integer values 2 and 3

$\nu = 2$: α,β etc. assume integer values 3 and 1 (2.1)

$\nu = 3$: α,β etc. assume integer values 1 and 2.

(iii) Summation Convention:

In this paper the summation convention on repeated indices is implied when the same index appears both as a lower and as an upper index. Thus the summation convention is implied in T^{α}_{α} but <u>not</u> in $T_{\alpha\alpha}$. The summation convention is also suspended when one repeated index is enclosed in the parentheses, e.g., as in $T^{(\alpha)}_{\alpha}$.

III. THE MATHEMATICAL MODEL

The mathematical development of the model equations used in this paper has already been published in [5], [6], [7]. However, for the sake of clarity of exposition we list here only the core steps leading to the final form of the equations. For the ensuing development we shall constantly use the conventions and symbols as stated in Section II of the paper.

The formulae of Gauss, (cf. [4], [17]) for a surface ν = const. are written compactly as

$$\underset{\sim}{r}_{,\alpha\beta} = \Gamma^{\delta}_{\alpha\beta}\underset{\sim}{r}_{,\delta} + \underset{\sim}{n}^{(\nu)} b_{\alpha\beta} \tag{3.1}$$

Inner multiplication of (3.1) by $G_\nu\, g^{\alpha\beta}$ then yields

$$D\underset{\sim}{r} + G_\nu (\Delta_2^{(\nu)}\, x^\delta)\, \underset{\sim}{r}_{,\delta} = \underset{\sim}{n}^{(\nu)} R, \tag{3.2}$$

where

$$D = G_\nu\, g^{\alpha\beta} \partial_{\alpha\beta}, \tag{3.3}$$

and

$$R = G_\nu g^{\alpha\beta}\, b_{\alpha\beta} = G_\nu\, (k_I^{(\nu)} + k_{II}^{(\nu)}). \tag{3.4}$$

Equation (3.2) provides three second-order partial differential equations for the determination of the Cartesian coordinates x_1, x_2, x_3 or x, y, z. However, before we impose any restriction on the Beltramians $\Delta_2^{(\nu)} x^\delta$ for the purpose of coordinate control, it is instructive to state the following results:

(i) All the terms in Eq. (3.2) depend only on the surface coordinates x^α. (Refer to the scheme (2.1) for the variation of α with ν).

(ii) For any allowable coordinate transformation $\bar{x}^\alpha$, i.e.,

$$x^\alpha \rightarrow \bar{x}^\alpha,$$

the *form* of the equation (3.2) remains invariant, i.e.,

$$\bar{D}\underset{\sim}{r} + \bar{G}_\nu(\bar{\Delta}_2^{(\nu)}\bar{x}^\delta)\ \underset{\sim}{r}_{,\delta} = \bar{\underset{\sim}{n}}^{(\nu)}\bar{R} \tag{3.5}$$

where now

$$\underset{\sim}{r}_{,\delta} = \frac{\partial \underset{\sim}{r}}{\partial \bar{x}^\delta} \quad \text{etc.,}$$

$$\bar{R} = \bar{G}_\nu(\bar{k}_I^{(\nu)} + \bar{k}_{II}^{(\nu)}).$$

(iii) It is important to note that both $\underset{\sim}{n}^{(\nu)}$ and $k_I^{(\nu)} + k_{II}^{(\nu)}$ (twice the mean curvature of a surface) are coordinate invariants, viz.,

$$\underset{\sim}{n}^{(\nu)} = \bar{\underset{\sim}{n}}^{(\nu)}, \tag{3.6a}$$

$$k_I^{(\nu)} + k_{II}^{(\nu)} = \bar{k}_I^{(\nu)} + \bar{k}_{II}^{(\nu)}. \tag{3.6b}$$

From Eq. (3.4),

$$k_I^{(\nu)} + k_{II}^{(\nu)} = g^{\alpha\beta}\, b_{\alpha\beta}, \tag{3.7}$$

which is the intrinsic form of the mean curvature. To have an extrinsic form, we consider the formula of the second derivative of the position vector $\underset{\sim}{r}$ in a 3D Euclidean space given as, e.g., [7],

$$\underline{r}_{,ij} = \Gamma^k_{ij}\underline{r}_{,k} \tag{3.8}$$

For the surface x^ν = const., Eq. (3.8) is rewritten as

$$\underline{r}_{,\alpha\beta} = \Gamma^k_{\alpha\beta}\ \underline{r}_{,k}\ , \tag{3.9}$$

where the derivatives with respect to x^ν appearing in $\Gamma^k_{\alpha\beta}$ are assumed to have been evaluated at x^ν = const. Taking the dot product of (3.9) with $\underline{n}^{(\nu)}$ and observing that $\underline{n}^{(\nu)}$ is orthogonal to any two vectors among $\underline{r}_{,k}$, we get

$$\underline{n}^{(\nu)} \cdot \underline{r}_{,\alpha\beta} = \Gamma^\nu_{\alpha\beta}\ \lambda^{(\nu)} \tag{3.10}$$

where

$$\lambda^{(\nu)} = \underline{n}^{(\nu)} \cdot \underline{r}_{,\nu}$$

Thus, the extrinsic form of (3.7) is

$$k_I^{(\nu)} + k_{II}^{(\nu)} = g^{\alpha\beta}\Gamma^\nu_{\alpha\beta}\ \lambda^{(\nu)} \tag{3.11}$$

By using the form (3.11) in Eq. (3.2), we get the model equations for the generation of another surface from the data of x, y, z, $x_{,\nu}$, $y_{,\nu}$, $z_{,\nu}$ of a given surface. This scheme eventually forms a method for 3D coordinate generation between any two (or more) given surfaces. For more details and computational results on this aspect of the use of Eq. (3.2) refer to [7], [8], [9], [16].

Since the subject matter of the present paper is confined to the problem of generation of surface coordinates in a <u>given</u> surface, it is sufficient for us to keep the intrinsic form (3.7) in Eq. (3.2).

IV. THE GENERATING SYSTEM

The model equation for generating the coordinates in a given surface is now taken as the equation (3.2). To be specific, we take the surface x^3 = const. (i.e., $\nu = 3$) as the given surface. In expanded form, taking $\underset{\sim}{r} = (x,y,z)$, $x^1 = \xi$, $x^2 = \eta$, $x^3 = \zeta$, the three equations are

$$Dx + [(\Delta_2^{(3)}\xi)\, x_\xi + (\Delta_2^{(3)}\eta)\, x_\eta]G_3 = X^{(3)}R, \tag{4.1}$$

$$Dy + [(\Delta_2^{(3)}\xi)\, y_\xi + (\Delta_2^{(3)}\eta)\, y_\eta]G_3 = Y^{(3)}R, \tag{4.2}$$

$$Dz + [(\Delta_2^{(3)}\xi)\, z_\xi + (\Delta_2^{(3)}\eta)\, z_\eta]\, G_3 = Z^{(3)}R, \tag{4.3}$$

where

$$D = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta}, \tag{4.4}$$

and

$$\Delta_2^{(3)}\xi = [\frac{\partial}{\partial\xi}(g_{22}/\sqrt{G_3}) - \frac{\partial}{\partial\eta}(g_{12}/\sqrt{G_3})]/\sqrt{G_3} \tag{4.5}$$

$$= (2g_{12}\Gamma^1_{12} - g_{11}\Gamma^1_{22} - g_{22}\Gamma^1_{11})/G_3,$$

$$\Delta_2^{(3)}\eta = [\frac{\partial}{\partial\eta}(g_{11}/\sqrt{G_3}) - \frac{\partial}{\partial\xi}(g_{12}/\sqrt{G_3})]/\sqrt{G_3}$$

$$=(2g_{12}\Gamma^2_{12} - g_{11}\Gamma^2_{22} - g_{22}\Gamma^2_{11})/G_3. \tag{4.6}$$

We now suitably restrict the distribution of the Beltramians so as to have available a sort of flexibility in the choice of the coordinates ξ, η in the surface. The most general form which can be chosen (cf. [6], is

$$\Delta_2^{(3)}\xi = g^{\alpha\beta}P^1_{\alpha\beta}, \tag{4.7a}$$

$$\Delta_2^{(3)}\eta = g^{\alpha\beta}\, P^2_{\alpha\beta}, \tag{4.7b}$$

where $P^1_{\alpha\beta}$, $P^2_{\alpha\beta}$ are symmetric in α, β and represent six arbitrary chosen control functions. Using the summation convention in (4.7) and the formulae

$$g^{11} = g_{22}/G_3,\; g^{12} = -\, g_{12}/G_3,\; g^{22} = g_{11}\,/G_3$$

we have

$$G_3\;\Delta_2^{(3)}\xi = \overline{P}\;,\; G_3\Delta_2^{(3)}\eta = \overline{Q}, \tag{4.8}$$

where

$$\overline{P} = g_{22}P^1_{11} - 2g_{12}P^1_{12} + g_{11}P^1_{22}, \tag{4.9a}$$

$$\overline{Q} = g_{22}P^2_{11} - 2g_{12}P^2_{12} + g_{11}P^2_{22}. \tag{4.9b}$$

The final form of the eqs. (4.1) - (4.3) is now

$$Lx = X^{(3)}R\;, \tag{4.10}$$

$$Ly = Y^{(3)}R\;, \tag{4.11}$$

$$Lz = Z^{(3)}R\;, \tag{4.12}$$

where

$$L = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta} +$$

$$G_3[(\Delta_2^{(3)}\xi)\partial_\xi + (\Delta_2^{(3)}\eta)\partial_\eta] \tag{4.13a}$$

$$= g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta} + \bar{P}\partial_{\xi} + \bar{Q}\partial_{\eta} \tag{4.13b}$$

Equations (4.10) - (4.13) are the basic generating equations for the curvilinear coordinates in a given surface. The function R appearing on the right hand side of the Eqs. (4.10)- (4.12) is given by

$$R = (k_I^{(3)} + k_{II}^{(3)})G_3.$$

The quantity in the parentheses is twice the mean curvature of the surface and as noted in Sect. III is invariant to the coordinates introduced is the given surface. Consequently $k_I^{(3)} + k_{II}^{(3)}$ reflects a basic geometrical aspect of the surface and is a function of the coordinates x, y, z. Since the Cartesian form of the equation $z = f(x,y)$ is assumed to be available for the surface under consideration, it is obvious from elementary differential geometry that

$$k_I^{(\nu)} + k_{II}^{(\nu)} = [(1 + z_y^2)\, z_{xx} - 2z_x\, z_y z_{xy} + (1 + z_x^2)\, z_{yy}]/(1 + z_x^2 + z_y^2)^{3/2}. \tag{4.14}$$

For arbitrary surfaces, it is always possible to generate the Cartesian form $z = f(x,y)$ of the surface by least square data fitting thus having $k_I + k_{II}$ as a function of x and y. It is also possible to solve only Eqs. (4.10) and (4.11) while calculating z from the given equation $z = f(x,y)$ of the surface. All these aspects have been discussed in Sect. VI.

V. GENERATING EQUATIONS BASED ON THE VARIATIONAL PRINCIPLE

Two attractive features of the chosen equations (4.10) - (4.13) are their simplicity and their explicit dependence on a basic surface- geometric property, viz., the mean curvature. Any other ad hoc, though consistent, set of equations can also be used to generate the coordinates but then it will be extremely difficult to isolate those basic surface-geometric properties which distinguish one surface from the other. In this connection the variational approach is also a possibility which has been used by Brackbill et al [19] for coordinate generation in a flat space, i.e., a 2D plane.

Generally, let us consider the surface functional

$$I = \int \sqrt{G_3}\, \phi \, d\xi d\eta. \tag{5.1}$$

Then $\delta I = 0$ leads one to the Euler-Lagrange equations

$$\frac{\partial}{\partial x_r} (\sqrt{G_3} \phi) - \frac{\partial}{\partial x^\beta} \frac{\partial(\sqrt{G_3}\phi)}{\partial x_{r,\beta}} = 0 \quad . \tag{5.2}$$

where, referring to Sect. II we have used the summation convention, x_r (r = 1,2,3) as the rectanguler Cartesian coordinates, and

$$x_{r,\beta} = \frac{\partial x_r}{\partial x^\beta}, \quad x_{r,\alpha\beta} = \frac{\partial^2 x_r}{\partial x^\alpha \partial x^\beta} \quad .$$

With these notations, it is a direct algebraic problem to show that

$$Lx_r = \frac{\sqrt{G_3}}{2} \frac{\partial}{\partial x^\gamma} \left(\frac{1}{\sqrt{G_3}} \frac{\partial G_3}{\partial x_{r,\gamma}}\right) , \tag{5.3}$$

where the operator L has earlier been defined in (4.13a). Let ϕ be a function only of $x_{r,\beta}$, then expanding (5.2) while using (5.3), we get

$$Lx_r = -M \quad , \tag{5.4}$$

where

$$M = \frac{\phi}{2}\left(\frac{\partial G_3}{\partial x_{r,\beta}}\frac{\partial \phi}{\partial x^\beta} + \frac{\partial \phi}{\partial x_{r,\beta}}\frac{\partial G_3}{\partial x^\beta}\right) + \frac{G_3}{\phi}\frac{\partial}{\partial x^\beta}\left(\frac{\partial \phi}{\partial x_{r,\beta}}\right) \tag{5.5}$$

Thus the generating system (5.4) is of the same form as originally proposed but it looks to be a formidable problem to select that ϕ which yields the right hand sides of Eqs. (4.10)-(4.13). One simple result for the case when $\phi = 1$ is obvious. For, in this case the minimization of I in (5.1) implies

$$Lx_r = 0 \quad .$$

and these are the equations for a minimal surface. Another case in which $\phi = F/G_3$ with F still as a function only of $x_{r,\beta}$ yields the Euler-Lagrange equations as (5.4) with M defined as

$$M = \frac{1}{2F}\left(\frac{\partial G_3}{\partial x_{r,\beta}}\frac{\partial F}{\partial x^\beta} + \frac{\partial G_3}{\partial x^\beta}\frac{\partial F}{\partial x_{r,\beta}}\right)$$

$$- \frac{1}{2G_3}\frac{\partial G_3}{\partial x_{r,\beta}}\frac{\partial G_3}{\partial x^\beta} - \frac{G_3}{F}\frac{\partial}{\partial x^\beta}\left(\frac{\partial F}{\partial x_{r,\beta}}\right) = 0 \tag{5.6}$$

The above case of $\phi = F/G_3$ is of interest because the choice

$$F = g_{11} + g_{22}, \tag{5.7a}$$

or

$$\phi = g^{11} + g^{22} \tag{5.7b}$$

is equivalent to the "smoothness" problem in 2D plane coordinates as shown by Brackbill [19]. It must, however, be stated that "smoothness" of coordinates in a 2D plane problem is due to the satisfaction of the Laplace equations. No such criteria is obvious by using (5.7) in (5.6), though it will of course yield a consistent set of three equations for the determination of x_1, x_2, x_3.

VI. NUMERICAL IMPLEMENTATION

The numerical solution of Eqs. (4.10)-(4.12) can be obtained by any suitable numerical method of solution which has proved useful in any elliptic grid generation problem. In this paper the equations have been discritized by using central differences for both the first and second derivatives and then solved iteratively from an initial guess by using the LSOR. The main difference between the coordinates in a flat space and in a surface is the appearance of the right hand side terms in which the quantity R can be established a' rpriori. This requires a knowledge of the equation of the surface $z = f(x,y)$, which when used in (4.14) yields R as a function of x and y. For arbitrary surfaces the equation $z = f(x,y)$ can be established by the least square method, [18].

To demonstrate the potential of Eqs. (4.10)-(4.12) as a viable set of equations for the generation of surface coordinates, we have selected three well known surfaces for the purpose of introducing a desired system of coordinates in them.

a. Coordinates In An Elliptic Cylinder Forming A Simply-connected Domain.

This problem is a prototype of coordinate generation in a given piece of a surface. The region under consideration forms a simply-connected region bounded by the space arc $\eta = \eta_0, \eta = \eta_1$, $\xi = \xi_0$, and $\xi = \xi_1$. Here $\eta = \eta_0, \eta_1$ are the elliptical arcs in the xy-plane, and $\xi = \xi_0, \xi_1$, are straight-lines parallel to the z-axis.

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a. Coordinates In An Elliptic Cylinder Forming A Simply-connected Domain.

This problem is a prototype of coordinate generation in a given piece of a surface. The region under consideration forms a simply-connected region bounded by the space arc $\eta = \eta_o, \eta = \eta_1$, $\xi = \xi_o$, and $\xi = \xi_1$. Here $\eta = \eta_o, \eta_1$ are the elliptical arcs in the xy-plane, and $\xi = \xi_o, \xi_1$, are straight-lines parallel to the z-axis.

The equations are:

$$\eta = \eta_o : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ z = z_o,$$

$$\eta = \eta_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ z = z_1, \tag{6.1}$$

$$\xi = \xi_o : x = -a, \ y = 0, \ z_1 \leq z \leq z_o,$$

$$\xi = \xi_1 : x = a, \ y = 0, \ z_1 \leq z \leq z_o.$$

The Dirichlet boundary conditions are provided by the data of (6.1) for the solution of Eqs. (4.10)-(4.12).

b. Coordinates In An Arbitrary Ellipsoid Cut By The Planes $z = z_o$, $z = z_1$.

This case is of coordinate generation in a doubly-connected region bounded by two closed space curves on an ellipsoid. The space curves $\eta = \eta_o$ and $\eta = \eta_1$, are given by

$$\eta = \eta_o : \frac{x^2}{\alpha^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \ z = z_o,$$

$$\eta = \eta_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \ z = z_1. \tag{6.2}$$

We now imagine a cut joining the curves $\eta = \eta_o$, $\eta = \eta_1$ while still remaining in the surface. As in the 2D case no boundary conditions can be prescribed on the cut line. However, since the values of x,y,z above and below the cut should be the same, we impose the periodicity conditions:

$$x(\xi_1,\eta) = x(\xi_o,\eta),\ y(\xi_1,\eta) = y(\xi_o,\eta),\ z(\xi_1,\ \eta) = z(\xi_o,\eta). \quad (6.3)$$

The Dirichlet conditions (6.2) and the periodicity conditions (6.3) yield a unique solution to the set of equations (4.10) - (4.12).

C. Coordinates In An Arbitrary Elliptic Paraboloid Cut By The Plane $z = z_o,\ z = z_1$.

This is again the case of coordinate generation in a doubly-connected region bounded by two closed space curves on an elliptic paraboloid. The space curves $\eta = \eta_o$ and $\eta = \eta_1$, are given by

$$\eta = \eta_o\ :\ \frac{x^2}{\alpha} + \frac{y^2}{b^2} = z\ ,\ z = z_o,$$

$$\eta = \eta_1\ :\ \frac{x^2}{\alpha^2} + \frac{y^2}{b^2} = z\ ,\ z = z_1. \qquad (6.4)$$

Under the boundary data (6.4) along with the periodicity conditions (6.3) the equations (4.10) - (4.12) have been solved.

In all cases (a) - (c) the control functions $P^1_{\alpha\beta}$ and $P^2_{\alpha\beta}$ have been set equal to zero, i.e., $\Delta^{(3)}_2 \eta = 0$, $\Delta^{(3)}_2 \eta = 0$, and the results are demonstrated in Figs. 1 - 6 and Figs. 10 - 12. Figures 7 - 9 show the results of a coordinate concentration near the curve $z = z_o = 0.9$ of case (b). In this case we have taken

$$P^1_{\alpha\beta} = 0,\ P^2_{11} = P^2_{12} = 0,$$

and

$$P^2_{22} = -(2.0 + (\eta-\eta_o)\ln\kappa)\ln\kappa/(1.0 + (\eta-\eta_o)\ln\kappa), \tag{6.5}$$

where $\kappa = 1.1$ is a constant.

The computer programs which have been developed to solve the equations (4.10) - (4.12) have been used successfully for all the cases enumerated above both with and without coordinate contraction. Also each case was repeatd to determine whether it is necessary to solve the z-equation too along with the x and y-equation. It has been found that solving all the three equations (4.10) - (4.12) or solving only (4.10) and (4.11) while taking the z iteratively from z = f(x,y) produces almost the same results.

V. Conclusions

A set of second order partial differential equations have been developed and then solved numericaly to generate the coordinates in a given surface. The proposed equations are a logical outcome of the formulae of Gauss and thus explicity depend on some basic differential-geometric properties of the surface in which the coordinates are to be introduced. The proposed method of surface coordinate generation is simple to implement and is capable of extension to arbitrary surfaces.

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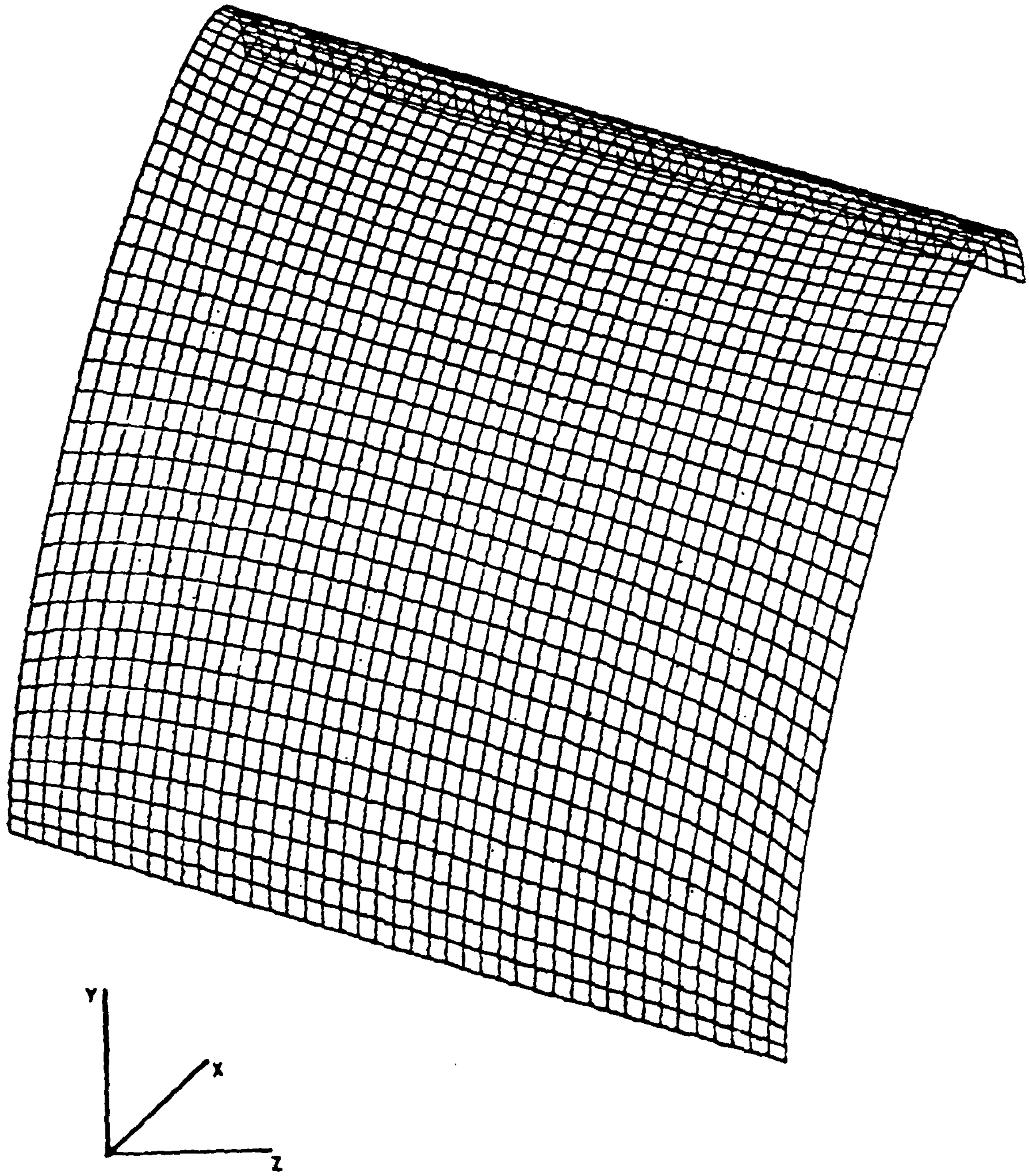

Figure 1. A simply-connected region on the surface of an elliptic cylinder; data: $a = 1.0$, $b = 0.5$, and $z = z_0 = 2.0$, $z = z_1 = 0.0$. Viewed after a 20° clockwise rotation about the x-axis.

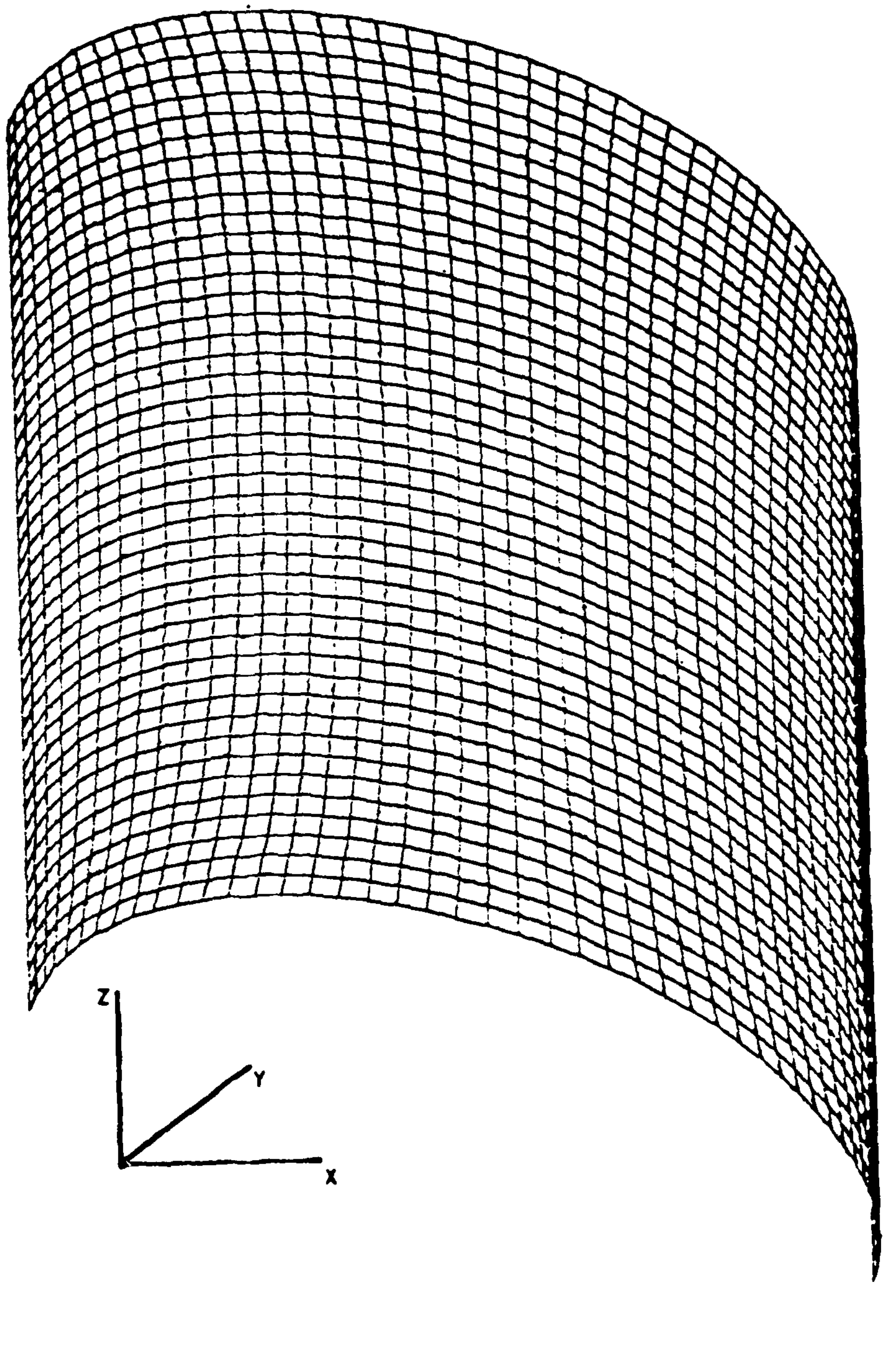

Figure 2. Data same as in Fig. 1. Viewed after a 20° clockwise rotation about the y-axis.

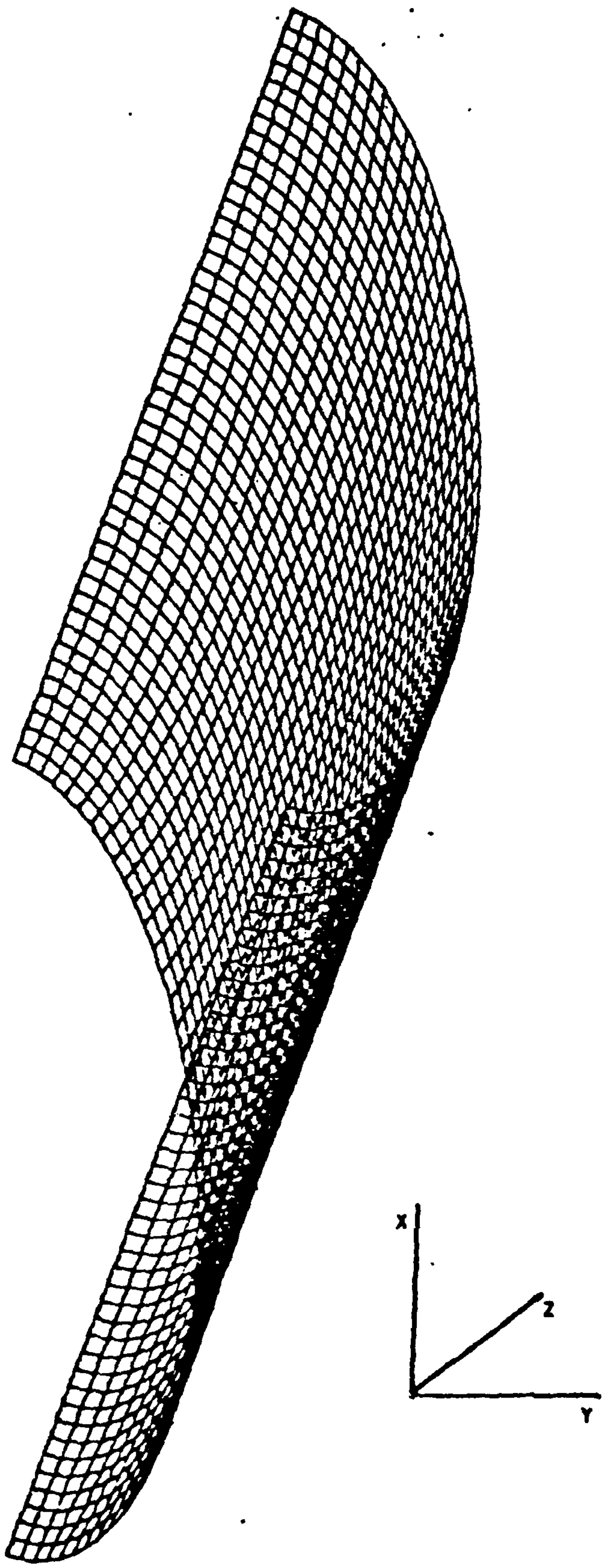

Figure 3. Data same as in Fig. 1. Viewed after a 20° clockwise rotation about the z-axis.

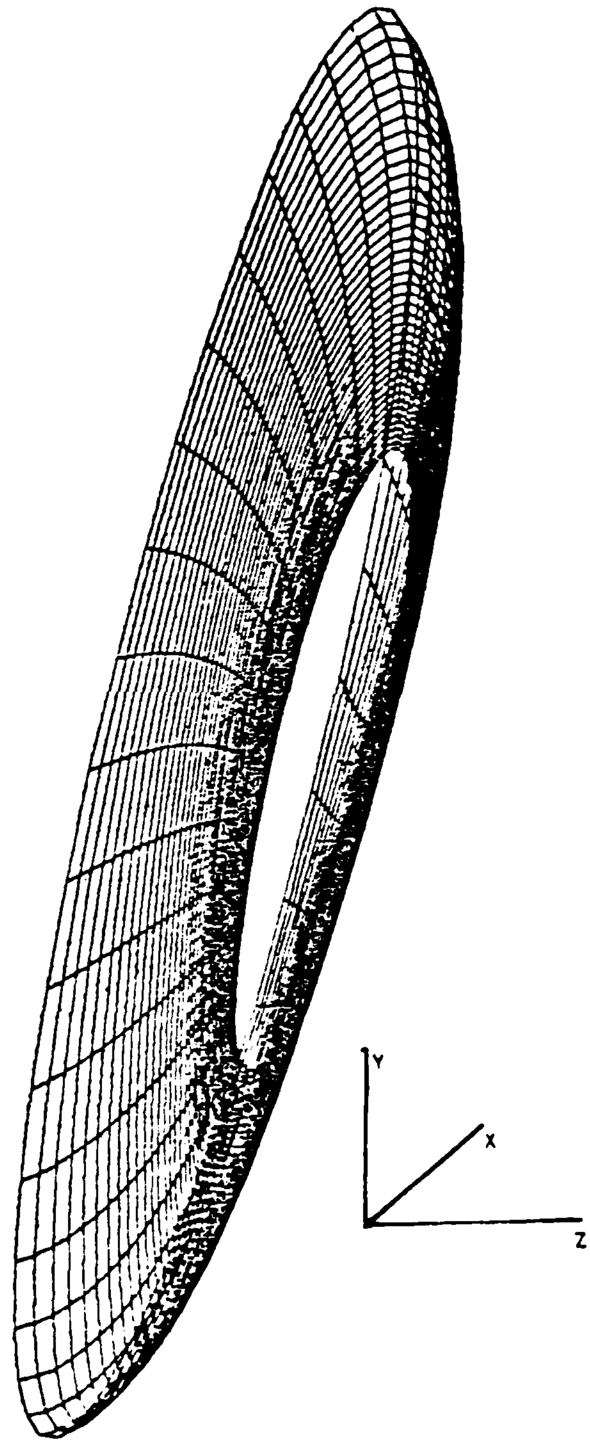

Figure 4. A doubly-connected region on the surface of an ellipsoid; data: $a = 5.0$, $b = 3.0$, $c = 1.0$, and cut by the planes $z = z_0 = 0.9$, $z = z_1 = 0$. Viewed after a 20° clockwise rotation abou the x-axis.

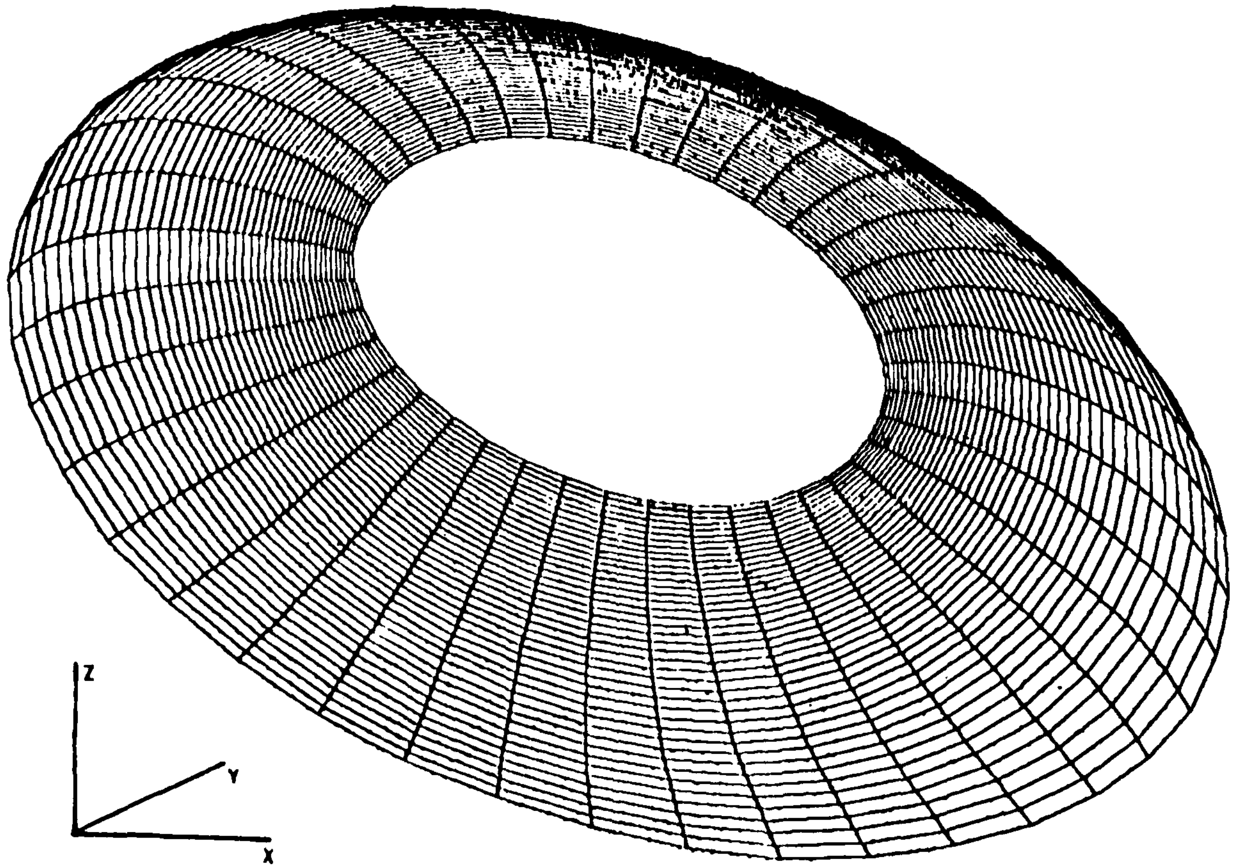

Figure 5. Data same as Fig. 4. Viewed after a 20° clockwise rotation about the y axis.

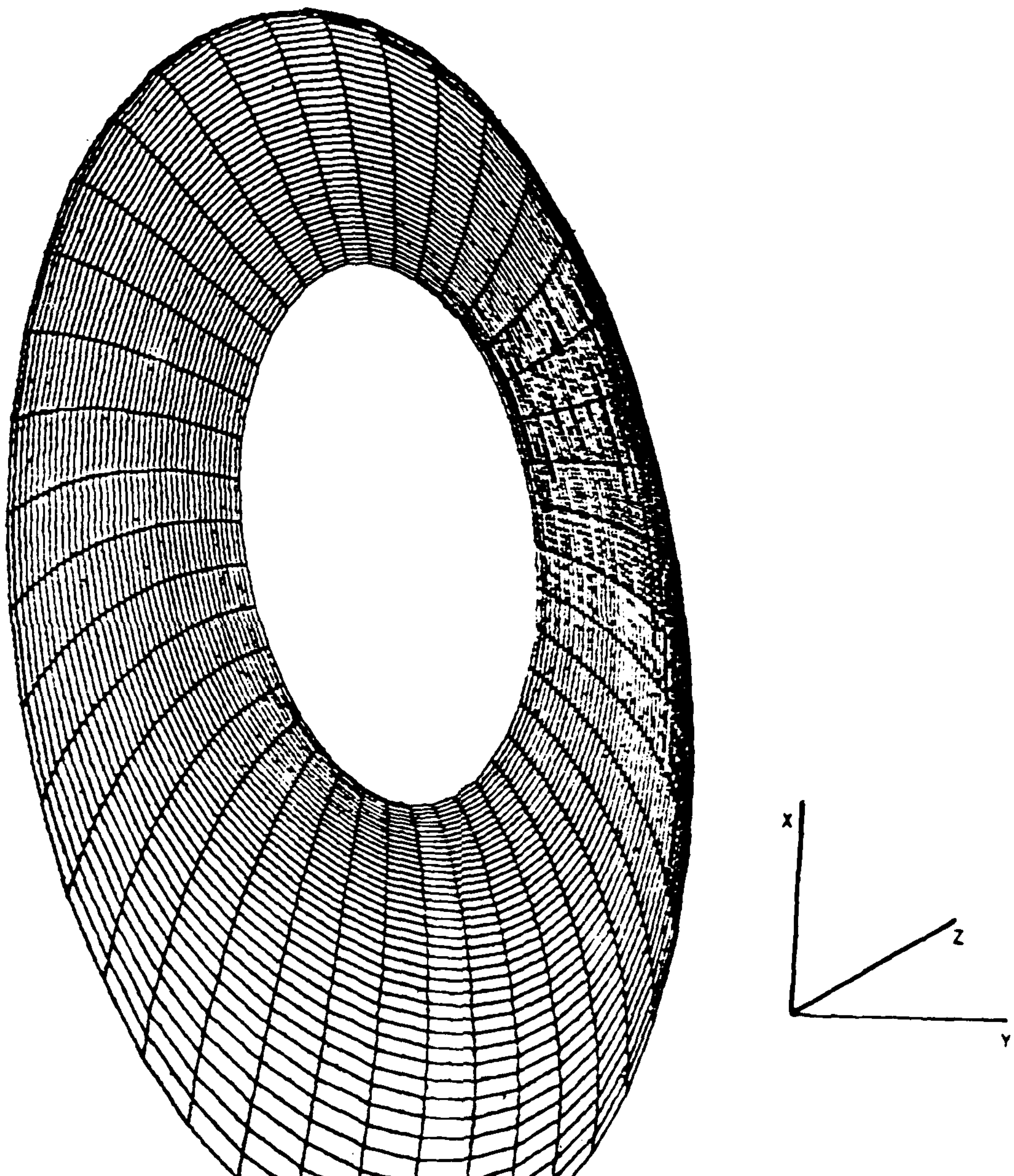

Figure 6. Data same as in Fig. 4. Viewed after a 20° clockwise rotation about the z-axis.

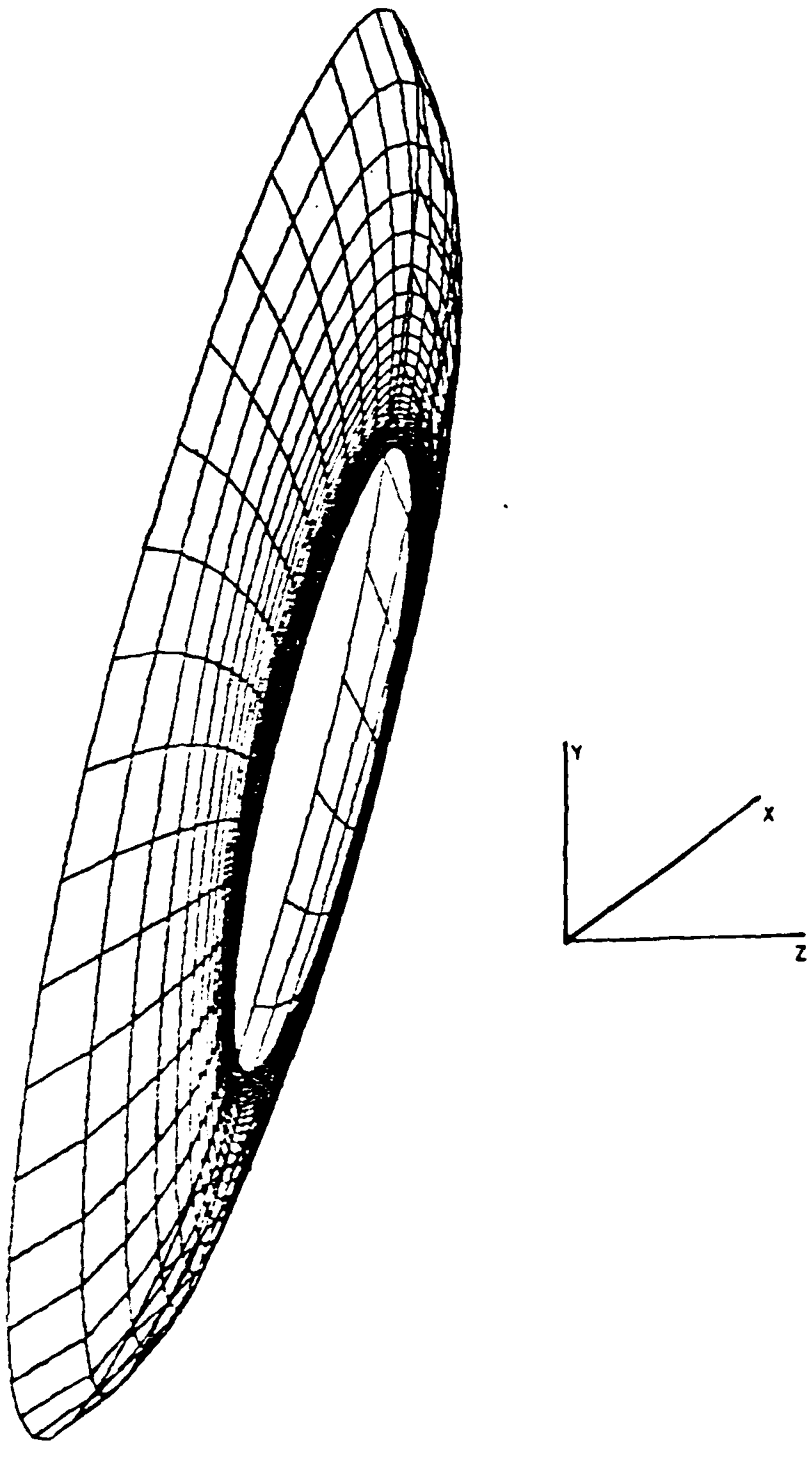

Figure 7. Data and configuration same as in Fig. 4, coordinate contraction around $z = z_0$.

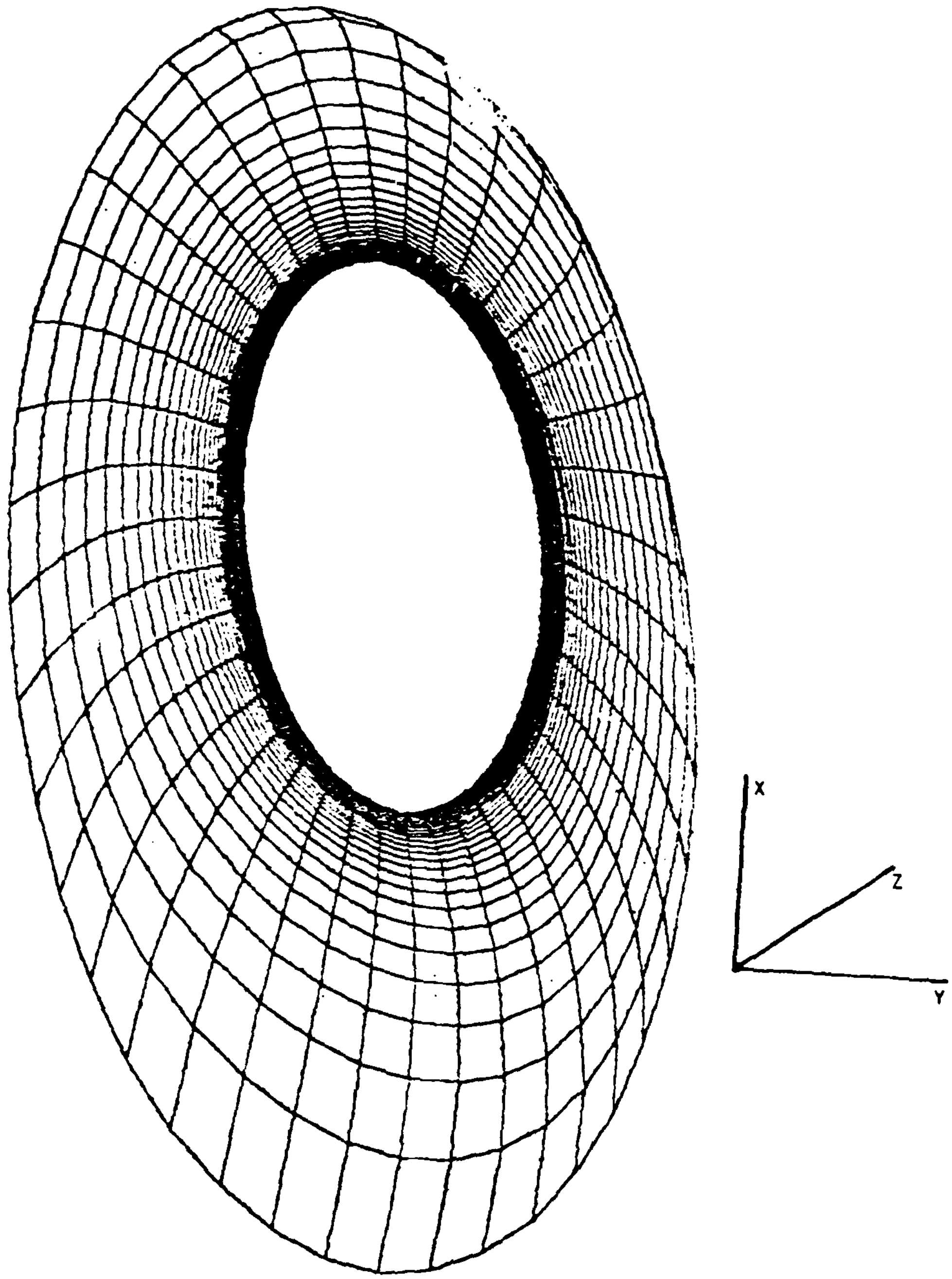

Figure 9. Data and configuration same as in Fig. 6, coordinate contraction near $z = z_0$.

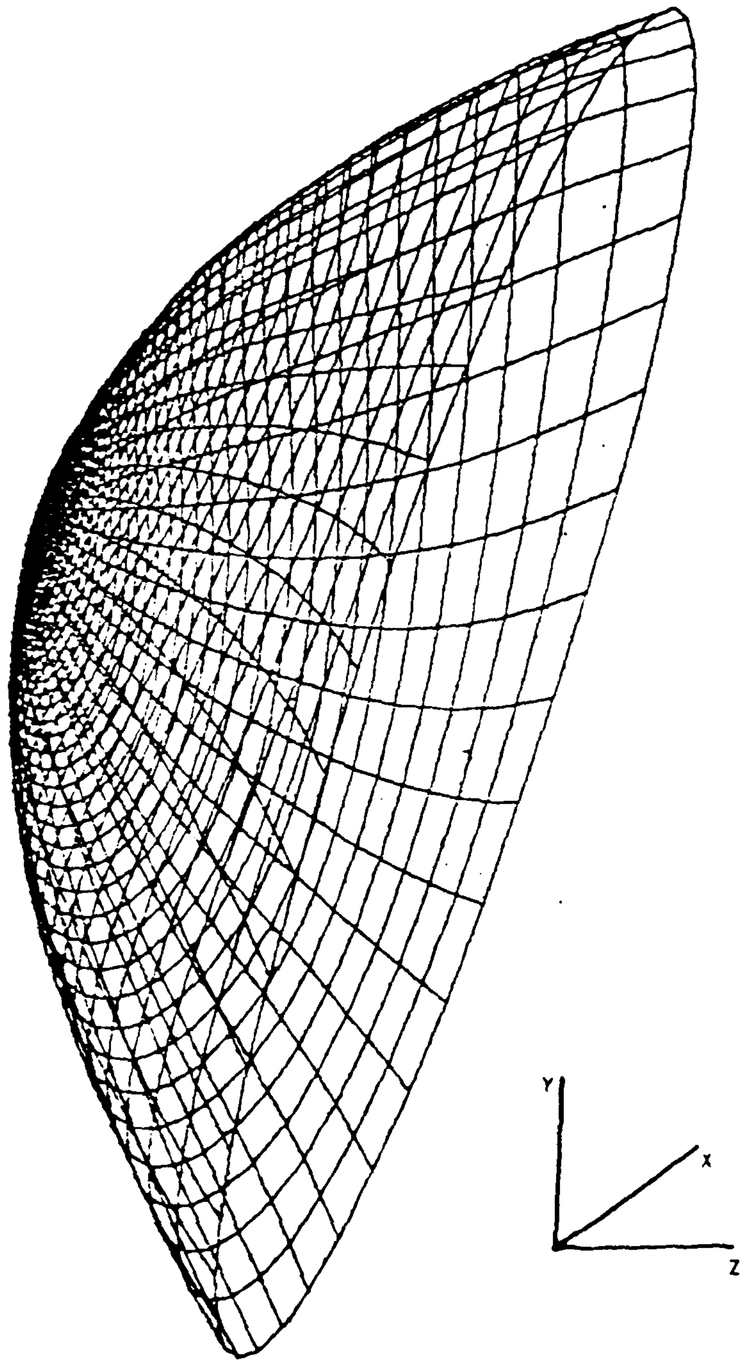

Figure 10. A doubly-connected region on the surface of an elliptic paraboloid; data: $a = 2.0$, $b = 1.0$, and cut by the planes $z = z_0 = 0.01$, $z = z_1 = 1.96$. Viewed after a 20° rotation about the x-axis.

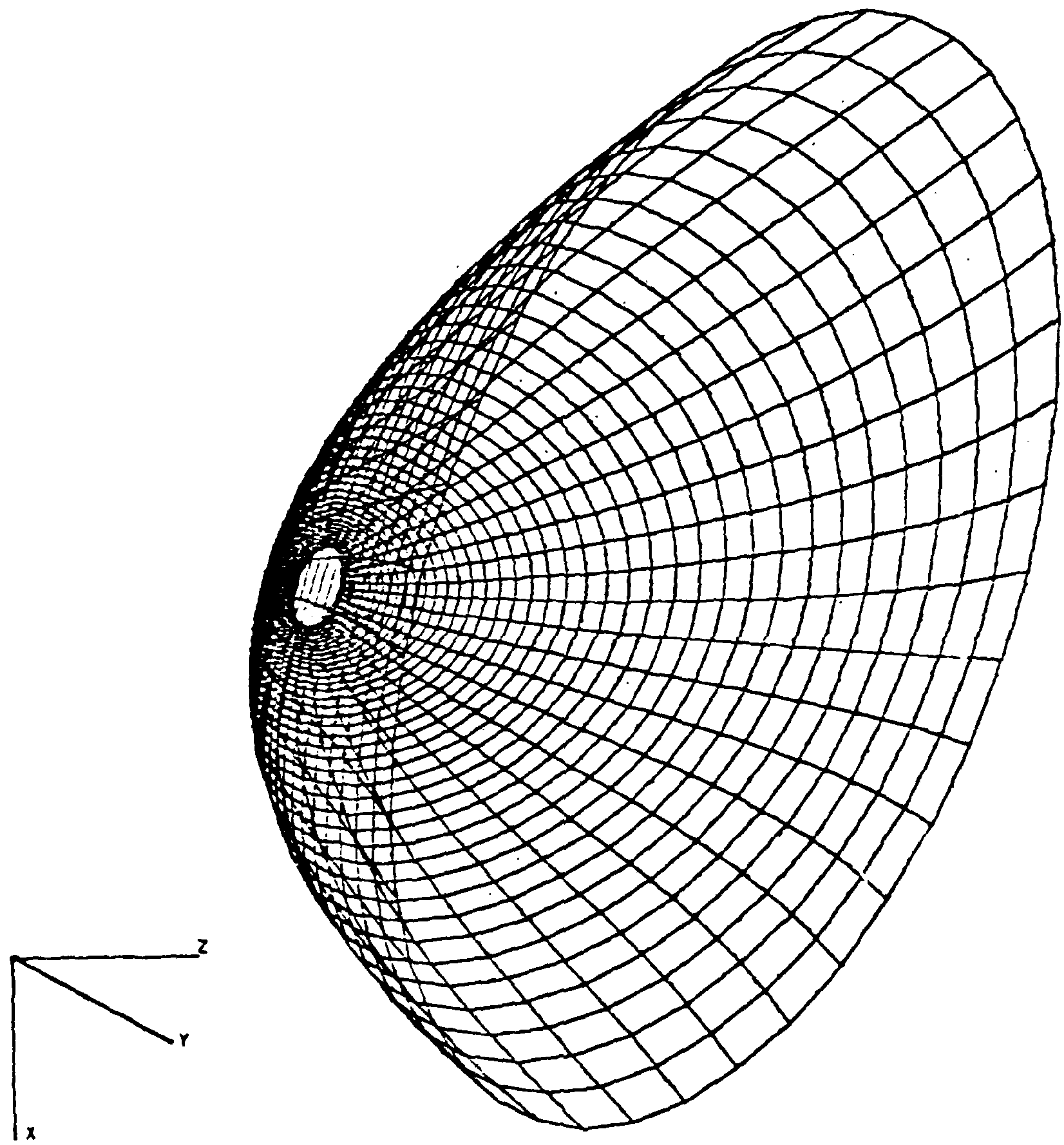

Figure 11. Data same as in Fig. 10. Viewed after a 20° clockwise rotation about the *y*-axis.

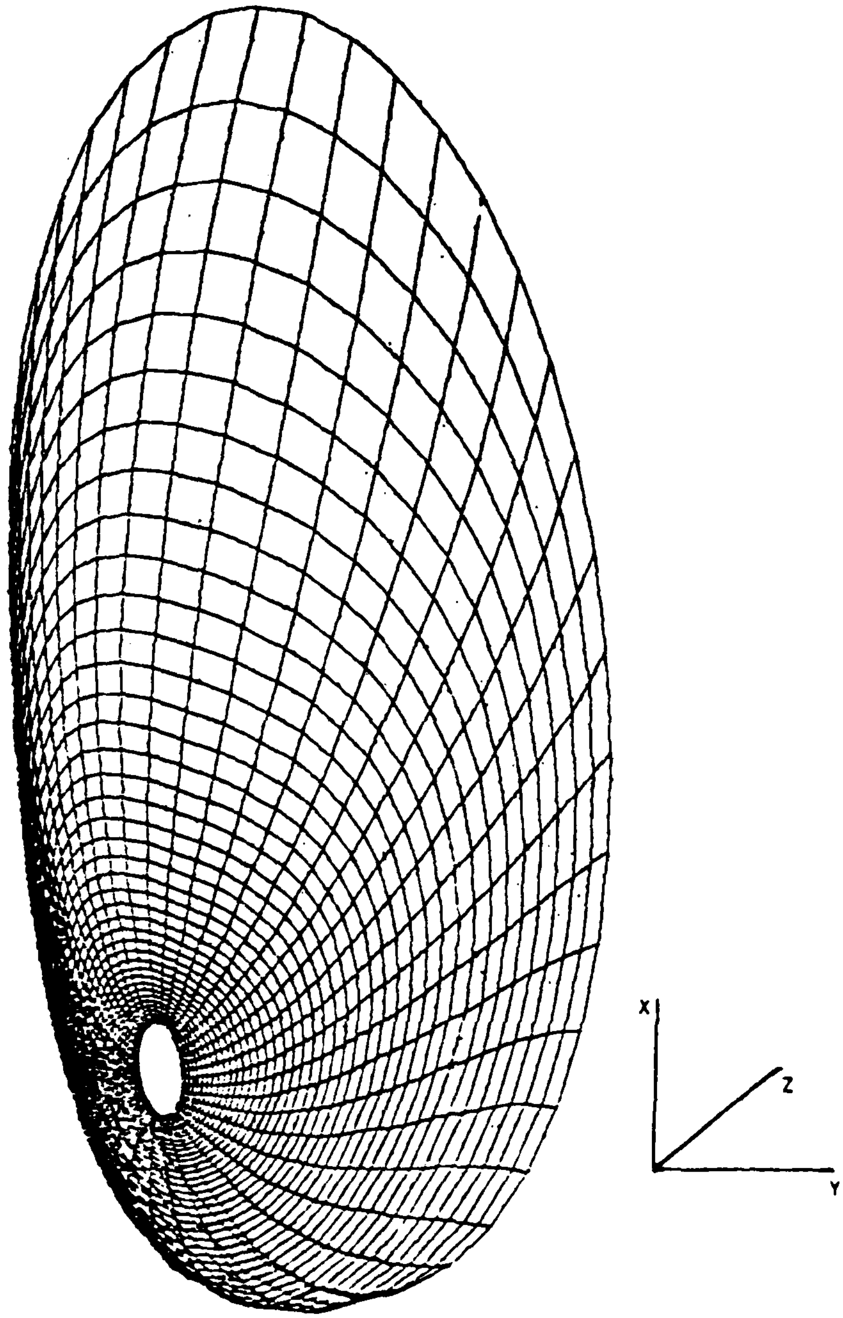

Figure 12. Data same as in Fig, 10. Viewed after a 20° clockwise rotation about the z-axis.

Computer simulation of three-dimensional grids

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ABSTRACT

This paper is devoted to the numerical solution of a set of second order elliptic partial differential equations for the generation of three-dimensional curvilinear coordinates between arbitrary shaped bodies. Starting from the given data on the inner and outer surfaces, a series of surfaces are generated which are connected with one another in such a manner that a sufficiently smooth three-dimensional coordinate net is realized. A number of cases pertaining to the numerical grid generation between two given surfaces have been discussed.

INTRODUCTION

Grid generation by computer simulation through carefully selected mathematical models has become a useful tool in many branches of engineering and physics. Although the developments in grid generation are mostly due to the problems in computational fluid dynamics, the techniques developed are also applicable to all those areas where field equations are to be solved on complicated domains. The main idea behind grid generation by computer simulation is to have a well structured mathematical model which is solved for the physical coordinates (the Cartesian coordinates) as functions of the curvilinear coordinates under the constraints of the specified boundaries on which the physical coordinates are known in advance.

In the generation of two- and three-dimensional coordinates around and between bodies of arbitrary shape there exists two main approaches: (i) algebraic methods (ii) methods based on solutions of partial differential equations (Preferably elliptic partial differential equations). The present method is concerned with the second class, that is, on solutions of elliptic partial differential equations. Refer to Thompson, Warsi, and Mastin (1) for more information on algebraic methods.

Elliptic grid generation systems have some advantages which seem to make them a better approach to take (2). The resulting grid is naturally smoother and there is less danger of overlapping of the grid lines. Because of the elliptic nature of the equations boundary slope discontinuities do not propagate into the field. Grids generated by solutions of elliptic equations are relatively easy to adapt to general boundary configurations and they have the ability to incorporate features such as concentration of grid lines, smoothness, and orthogonality.

This work is devoted to the generation of three-dimensional grids between two arbitrary shaped bodies. The method has its origin in the formulae of Gauss for a surface. In this scheme an inner and outer boundary surface is assumed to be initially given. The inner bodies investigated are those of various sizes of ellipsoids, fuselage shapes, two intersecting spheres or ellipsoids, and a wing-body combination. In all cases the outer surface is either a sphere or an ellipsoid. A series of surfaces are generated between the inner and outer boundary surfaces in such a way that only two coordinates vary in each surface. The equations solved (Eqs. (4)) are structured in such a way that there is an automatic connection between succeeding generated surfaces. The result is a sufficiently differentiable three-dimensional coordinate system in the enclosed region. Provisions are made for contraction of coordinates in the generated surfaces and results are shown for contraction near the inner body.

In the case of a wing-body combination being the inner body a set of partial differential equations (Eq. (6)) are solved which transform whatever coordinate system is initially given on the body to one in which one of the coordinate lines is the trace of the wing and body intersection. This new coordinate system allows for a smooth matching of the wing with the body. With this now as the coordinate system on the inner body the equations are solved to generate the three-dimensional coordinate system between the wing-body and the outer boundary surface.

MATHEMATICAL MODEL

The basic mathematical development of the method to be discussed is available in Warsi (3), (4), and the details on the computer simulation are available in Ziebarth (5), and Tiarn (6). Below we shall briefly state the essential structure of the proposed model.

As mentioned in the introduction, the present model has its origin in the formulae of Gauss for a surface. The formulae of Gauss are essentially a set of compatibility relations connecting the second derivatives with the surface Christoffel symbols, the coefficients of the second fundamental form, and the unit normal, all evaluated in the surface. Denoting the three Cartesian coordinates in space by $\underset{\sim}{r} = (x, y, z)$ and the curvilinear coordinates by x^i ($i = 1,2,3$), the formulae of Gauss for the surface $x^\nu = \text{const.}$ are

$$\underset{\sim}{r}_{\alpha\beta} = \Gamma^{\delta}_{\alpha\beta}\, \underset{\sim}{r}_{\delta} + b_{\alpha\beta}\, \underset{\sim}{n}^{(\nu)} , \qquad (1)$$

where repeated lower and upper indices imply summation, and

$$\underset{\sim}{r}_{\alpha\beta} = \frac{\partial^2 \underset{\sim}{r}}{\partial x^\alpha \partial x^\beta} , \quad \underset{\sim}{r}_{\delta} = \frac{\partial \underset{\sim}{r}}{\partial x^\delta} .$$

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In Eq. (1) $T^{\delta}_{\alpha\beta}$ are the surface Christoffel symbols defined as

$$T^{\delta}_{\alpha\beta} = \tfrac{1}{2} g^{\delta\sigma}\left(\frac{\partial g_{\alpha\sigma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\sigma}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\sigma}}\right).$$

where $g_{\alpha\beta}$ are the coefficients of the first fundamental form. The coefficients of the second fundamental form, $b_{\alpha\beta}$, are defined as

$$b_{\alpha\beta} = \underset{\sim}{n}^{(\nu)} \cdot \underset{\sim}{r}_{\alpha\beta} ,$$

where the unit normal $\underset{\sim}{n}^{(\nu)}$ on the surface x^{ν} = const. is

$$\underset{\sim}{n}^{(\nu)} = (\underset{\sim}{r}_{\alpha} \times \underset{\sim}{r}_{\beta})/|\underset{\sim}{r}_{\alpha} \times \underset{\sim}{r}_{\beta}| .$$

It must be mentioned that the range of the Greek indices is only on two values. Thus, for $\nu = 3$, the index range is from 1 to 2. From (1), *writing* the equations for $\alpha = 1$, $\beta = 1$, $\alpha = 1$, $\beta = 2$, and $\alpha = 2$, $\beta = 2$, and multiplying the equations *respectively by* g_{22}, g_{12} and g_{11} and adding them together, we get

$$D\underset{\sim}{r} + [(\Delta_2\xi)\underset{\sim}{r}_{\xi} + (\Delta_2\eta)\underset{\sim}{r}_{\eta}]G_3 = \underset{\sim}{n}R, \qquad (2)$$

where these equations are for the surface ζ = const. on which ξ and η are the current coordinates. In Eq. (2) a variable subscript implies a partial derivative, and

$$D = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta} ,$$

$$\Delta_2 = \frac{1}{\sqrt{G_3}}\left[\frac{\partial}{\partial\xi}\frac{1}{\sqrt{G_3}}(g_{22}\partial_{\xi} - g_{12}\partial_{\eta}) + \frac{\partial}{\partial\eta}\frac{1}{\sqrt{G_3}}(g_{11}\partial_{\eta} - g_{12}\partial_{\xi})\right] ,$$

$$G_3 = g_{11}g_{22} - (g_{12})^2 ,$$

$$\underset{\sim}{n} = (X,Y,Z) ,$$

$$R = (k_1 + k_2)G_3 ,$$

where $k_1 + k_2$ is twice the mean curvature of the surface at any point. The quantities $\Delta_2\xi$ and $\Delta_2\eta$ *are the surface Laplacians* or Beltramians.

A set of generating equations can now be obtained by setting

$$\begin{aligned} \Delta_2\xi &= Pg_{22}/G_3 , \\ \Delta_2\eta &= Qg_{11}/G_3 , \end{aligned} \qquad (3)$$

where P and Q are arbitrary specified functions which exert the desired control on the distribution of coordinates in the surface to be generated. Substituting (3) in (2), the elliptic partial differential equations for the generation of surface coordinates are

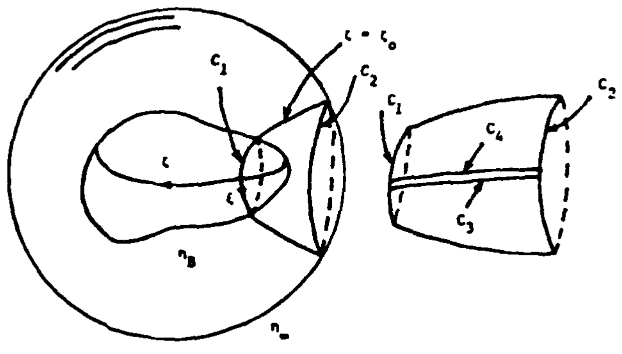
$$D\underset{\sim}{r} + Pg_{22}\underset{\sim}{r}_{\xi} + Qg_{11}\underset{\sim}{r}_{\eta} = \underset{\sim}{n}R . \qquad (4)$$

Referring to Fig. 1, let the given inner and outer body surfaces be given as $\eta = \eta_B$ and $\eta = \eta_{\infty}$ on which the values of $\underset{\sim}{r} = (x,y,z)$ are known. Since the coordinates ξ and η have been chosen in these surfaces, the derivatives $\underset{\sim}{r}_{\zeta}$ on the inner and outer surfaces. denoted as $(\underset{\sim}{r}_{\zeta})_B$ and $(\underset{\sim}{r}_{\zeta})_{\infty}$ are known. To connect one generated surface through Eq. (4) with the other *we specify*

$$\underset{\sim}{r}_{\zeta} = f_1(\eta)(\underset{\sim}{r}_{\zeta})_B + f_2(\eta)(\underset{\sim}{r}_{\zeta})_{\infty} \qquad (5)$$

where $f_1(\eta)$ and $f_2(\eta)$ are suitable weights which are such that

$$f_1(\eta_B) = 1,\ f_1(\eta_{\infty}) = 0,\ f_2(\eta_B) = 0,\ f_2(\eta_{\infty}) = 1.$$



(a) (b)

Fig. 1 (a) Topology of the given surfaces.
(b) Surface to be generated.

The set of Eqs. (4) essentially complete the development of a consistent mathematical model for the generation of surface coordinates, where the means of proceedings from one generated surface to the other is provided by Eqs. (5).

SURFACE COORDINATES

Equations (4) can also be used in a straight forward manner to yield surface coordinates based on prescribed data on certain contours of the surface itself. These equations, (7), provide a means of introducing a new coordinate system based on the data of an existing coordinate system, e.g., from a polar coordinate system of Craidon (8) to an O-type coordinate system in the surface, which may be needed in certain wing-body combinations. These equations are stated as follows.

$$a u_{\xi\xi} - 2b u_{\xi\zeta} + c u_{\zeta\zeta} + P u_\xi + Q u_\zeta = J_2^2 \Delta_2 u \, ,$$
$$a v_{\xi\xi} - 2b v_{\xi\zeta} + c v_{\zeta\zeta} + P v_\xi + Q v_\zeta = J_2^2 \Delta_2 v \, , \tag{6a}$$

where u, v are the parametric coordinates in the surface which are to be transformed to the new coordinates (ζ, ξ), and the coefficients a, b, etc., are

$$a = (\bar{g}_{11} v_\zeta^2 + 2\bar{g}_{13} u_\zeta v_\zeta + \bar{g}_{33} u_\zeta^2)/\bar{G}_2 \, ,$$
$$b = (\bar{g}_{11} v_\zeta v_\xi + \bar{g}_{13}(u_\zeta v_\xi + u_\xi v_\zeta) + \bar{g}_{33} u_\zeta u_\xi)/\bar{G}_2 ,$$
$$c = (\bar{g}_{11} v_\xi^2 + 2\bar{g}_{13} u_\xi v_\xi + \bar{g}_{33} u_\xi^2)/\bar{G}_2 \, ,$$
$$J_2 = \sqrt{\bar{G}_2} = u_\zeta v_\xi - u_\xi v_\zeta \, , \tag{6b}$$

and $\Delta_2 u$, $\Delta_2 v$ are the Beltramians defined earlier. An overhead bar means a metric coefficient in the u, v system, e.g.

$$\bar{g}_{11} = x_u^2 + y_u^2 + z_u^2 \qquad \text{etc.}$$

Equations (6a) and (6b) have been programmed in Refs. (5) and (6). In (5) these equations have been used to provide those coordinates which are such that the inner 0-type coordinate conforms to the contour created by the intersection of a wing with the fuselage. Equations similar to Eqs. (6a) and (6b) have been derived by Garon and Camarero (9) and Whitney and Thomas (10), the later reference has followed a very tedius procedure.

NUMERICAL SOLUTION OF 3-D EQUATIONS

The system of partial differential equations, (Eqs. (4), for the generation of a three-dimensional coordinate system between two arbitrary shaped bodies has been solved by finite differences using point-SOR. Initially the coordinates on the two boundary surfaces must be prescribed by some method. In this research these coordinates were prescribed analytically for simple shapes or were calculated by a computer subroutine of Craidon (8). Although the choice of an outer boundary surface is arbitrary there must be a one-to-one correspondence between points on the two boundary surfaces. In addition, these points need to be chosen so that the lines joining the points on the inner and outer surfaces do not intersect.

Regardless of the method chosen to correspond points between the inner and outer boundary surfaces (see (5) for a discussion of these) it has been found to be a shape dependent mechanical exercise. For example, Figures 2 and 3 show the correspondence between points on the inner and outer surface for the case of intersecting ellipsoids. Although in both cases there is a one-to-one correspondence between points on each surface, when the correspondence of Fig. 2 is used the solution will not converge, while that of Fig. 3 does yield a solution to the differential equations. There are two main problems with the correspondence of Fig. 2. The lines near the right end are close together and appear to be parallel. This is believed to be the reason for the divergence of the solution since the computations stopped in this region. Another problem with the correspondence in Fig. 2 is that the correspondence line labeled A in Fig. 2 is not vertical. This line should be chosen to be along the y-axis (x = z = 0) so that it is not permitted to change from a quadrant where x is negative to one where x is positive. This second correspondence problem was found to cause even the solution, for a simple ellipsoid as the inner body, to diverge.

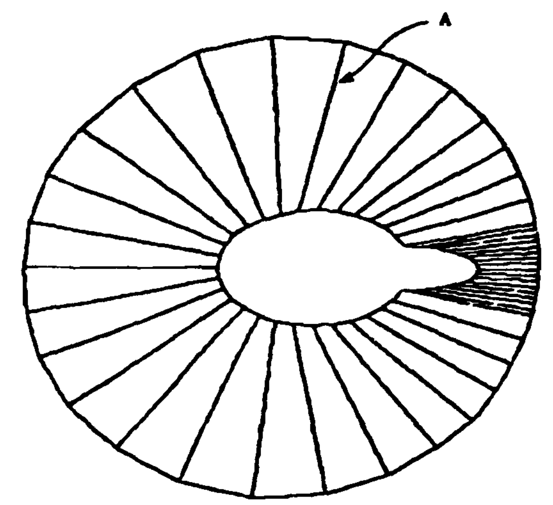


Fig. 2 Correspondence of points which does not allow convergence of the solution.

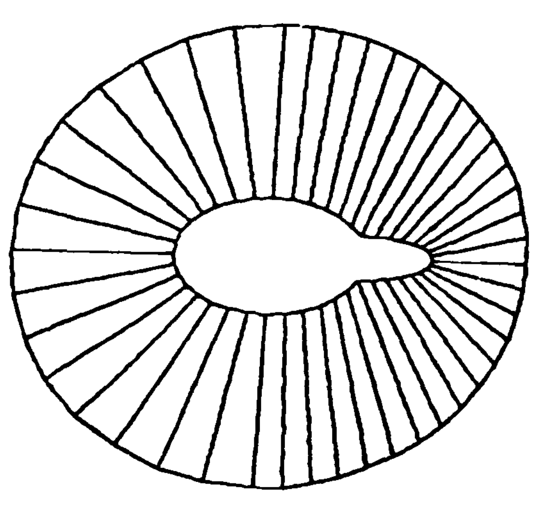


Fig. 3 Correspondence of points which allows convergence of the solution.

Runs were made with both an ellipsoid and a sphere as the outer body. No advantage was found in taking the outer body to be an ellipsoid so a sphere was used in all other cases.

In conclusion, the correspondence between the inner and outer body is important for the solution of equations (4) to converge. There does not seem to be a general method which works for all inner body shapes, therefore, the most useful way seems to be a geometric manipulation which is dependent on the inner body configuration.

Once the correspondence has been established between the inner and outer bodies an initial guess must be made for the values of the Cartesian coordinates in the field. In all cases linear interpolation between the points on the inner and outer surfaces was used successfully to accomplish this.

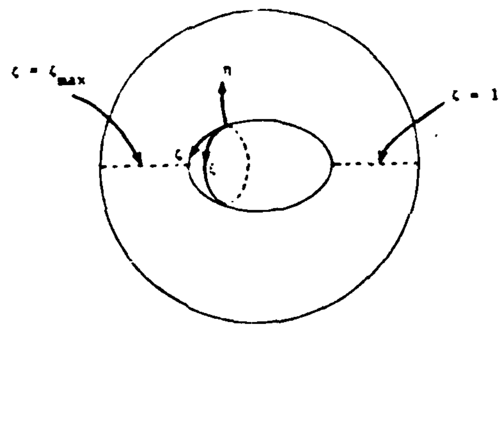


Fig. 4 Lines $\zeta = 1$ and $\zeta = \zeta_{max}$ for a prolate ellipsoid surrounded by a sphere.

Before a solution algorithm can be applied to equations (4) appropriate boundary conditions for $\zeta = 1$ and $\zeta = \zeta_{max}$ (Fig. 4) must be determined. For the case of a prolate ellipsoid surrounded by a sphere an exact solution exists and is used to calculate these boundary conditions. The solution is given by, (4),

$$x = [A\, e^{B\eta(\bar\eta)} + C]\cos\zeta \tag{7}$$

where

$$A = \frac{(e^{\eta_B} - \cosh\eta_B)\sinh\eta_B}{e^{\eta_B} - \sinh\eta_B}$$

$$B = \ln\{[\frac{e^{\eta_\infty}}{\sinh\eta_B}]^{\frac{1}{\eta_\infty - \eta_B}}\}$$

$$C = \frac{e^{\eta_\infty}(\cosh\eta_B - \sinh\eta_B)}{e^{\eta_\infty} - \sinh\eta_B}$$

$$D = \sinh\eta_B$$

and $\eta(\bar\eta)$ may be taken as

$$\eta(\bar\eta) = \frac{(\eta_\infty - \eta_B)(\bar\eta - \bar\eta_B)}{\bar\eta_\infty - \bar\eta_B}\,\kappa^{(\bar\eta - \bar\eta_\infty)}$$

where $\kappa = 1.0$ implies no contraction near the inner body and κ slightly greater than 1.0 (e.g. 1.05 or 1.20) implies contraction. Here $\bar\eta$ is enumerated as an integer.

In the case of more complicated shapes such as intersecting ellipsoids the exact solution may not be known between it and an outer surface. However, for certain lines in the field equations (4) can be solved in their limiting forms. One such line is the line A'B' in Fig. 5. Warsi (4) has shown that along this line the expression for x can be written as

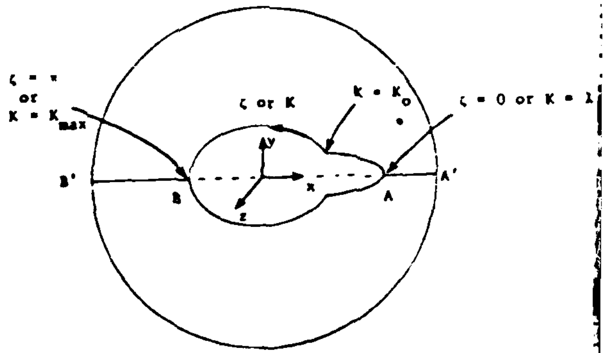


Fig. 5 Intersecting ellipsoids surrounded by a sphere.

$$x(\bar\eta) = x_B + \frac{x_\infty - x_B}{e - 1}(e^{S(\bar\eta)} - 1) \tag{8}$$

where

$$S(\bar\eta) = \frac{\bar\eta - \bar\eta_B}{\bar\eta_\infty - \bar\eta_B}\,\kappa^{\bar\eta - \bar\eta_\infty} \tag{9}$$

and the subscripts B and ∞ refer to the inner and outer body respectively. The expression for x in (8) has been used in the case of intersecting spheres and ellipsoids to evaluate the boundary conditions on the right segment (AA' in Figure 5). A value of κ slightly larger than 1.0 is required in (9) to produce coordinate systems which are smooth near the right segment. Because of the orientation of the intersecting bodies with the Cartesian coordinates the boundary conditions for the left segment (BB' in Figure 5) can be determined by an expression like (7).

Once the coordinates on the inner and outer bodies have been specified the ζ-derivatives on these surfaces can also be calculated by second order finite differences. These need to be calculated on both the inner and outer surfaces for all values of K from K = 2 to K = KMAX-1 and for all values of I from I = 1 to I = IMAX (Fig. 6). The ζ-derivatives are not required at K = 1 and K = KMAX since these are boundary conditions.

Special care must be taken at locations where there exists a mathematical discontinuity (e.g., at the intersection point $K = K_0$ in Fig. 5). A simple averaging of the ζ-derivatives through this region was found to adequately smooth out these derivatives.

Two more concerns need to addressed before the differential equations can be solved. The first is the specification of the functions $f_1(\eta)$ and $f_2(\eta)$ appearing in equation (5). In this research f_1 was taken to be a linear function of $\bar{z}$ where

$$\bar{z} = \frac{\eta_\infty - \eta}{\eta_\infty - \eta_B}$$

and $f_2(\bar{z}) = 1 - f_1(\bar{z})$. The second concern is the specification of the redistribution functions P and Q in equation (4). In this work only contraction of the η-lines near the inner body was considered, therefore, only the function Q was specified and P was set equal to zero. Based on earlier two-dimensional work (11), (12), Q was prescribed as

$$Q = \frac{\bar{g}_{11}[2 + (\bar{\eta} - \bar{\eta}_\infty)\ln\kappa]\ln\kappa}{1 + (\bar{\eta} - \bar{\eta}_B)\ln\kappa} .$$

NUMERICAL SOLUTION OF SURFACE EQUATIONS

In the case of a wing-body combination equations (4) are solved, but only after a transformation of the coordinate system on the body is made. The body is initially covered by a coordinate system (u,v). By initially solving equations (6) the (u,v) coordinate system is transformed into a (ξ,ζ) coordinate system on the same body. The (ξ,ζ) coordinate system will cover only half of the original body, but it has the advantage of having one of its coordinate lines as the trace of the intersection between the body and the wing, Fig. 7.

In order to solve the differential equations (6) boundary conditions need to be chosen on the surface of the body. Although the method for picking points is arbitrary they need to satisfy a few basic requirements. The inner boundary is to be the trace of the intersection line between the body and wing (line labeled A in Fig. 8). The outer boundary (line B in Fig. 8) should be very close to the line which cuts the body into two similar halfs (line C in Fig. 8). It can even be coincident with part of this curve as shown in Figure 8. However, the outer boundary should not be taken to be entirely on this line since it needs to have a variation over the surface of the body, i.e., a variation in all three Cartesian coordinates x, y and z.

When picking the points on the inner or outer boundary it is necessary to be able to find both the Cartesian coordinates (x,y,z) and the (u,v) coordinates for points within any patch. This is accomplished by considering parameters μ and ν which vary from 0 to 1 within any patch. Here μ varies in the direction of the u coordinate and ν varies in the direction of the v coordinate. Obviously since u

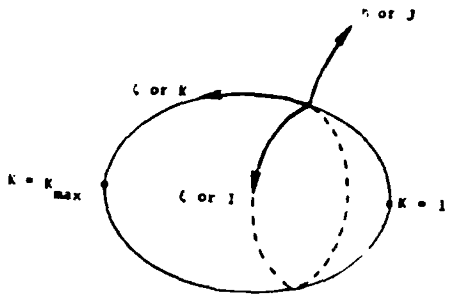


Fig. 6 Nomenclature for ellipsoid as an inner boundary shape.

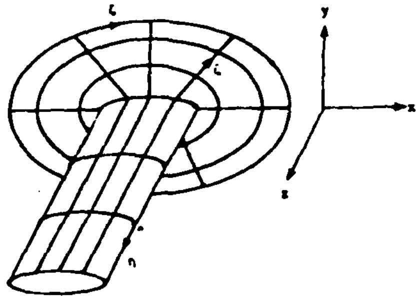


Fig. 7 Coordinates ζ, ξ, η on the wing-body combination.

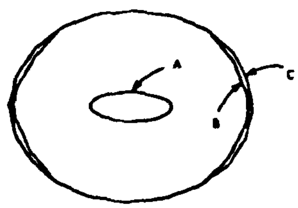


Fig. 8 Choice of boundary curves on inner body.

varies from 0 to π then

$$\frac{u}{\lambda} = \mu \qquad (10)$$

where

$$\lambda = \frac{\pi}{NPU}$$

NPU = number of patches in U direction

and

$$\frac{v}{\sigma} = \nu \qquad (11)$$

where

$$\sigma = \frac{\pi}{NPV}$$

NPV = number of patches in V direction

Based on the parameters μ and ν a blending equation can be defined for these surface patches similar to the one used by Craidon (8). This bicubic surface patch equation has the form

$$\begin{aligned} \underline{r}(\mu,\nu) = {} & F_1(\mu)F_1(\nu)\underline{r}_{0,0} + F_1(\mu)F_2(\nu)\underline{r}_{0,1} \\ & + F_1(\nu)F_2(\mu)\underline{r}_{1,0} + F_2(\mu)F_2(\nu)\underline{r}_{1,1} \\ & + F_1(\nu)F_3(\mu)\underline{r}_{\mu_{0,0}} + F_2(\nu)F_3(\mu)\underline{r}_{\mu_{0,1}} \\ & + F_1(\nu)F_4(\mu)\underline{r}_{\mu_{1,0}} + F_2(\nu)F_4(\mu)\underline{r}_{\mu_{1,1}} \\ & + F_1(\mu)F_3(\nu)\underline{r}_{\nu_{0,0}} + F_1(\mu)F_4(\nu)\underline{r}_{\nu_{0,1}} \\ & + F_2(\mu)F_3(\nu)\underline{r}_{\nu_{1,0}} + F_2(\mu)F_4(\nu)\underline{r}_{\nu_{1,1}} \\ & + F_3(\mu)F_3(\nu)\underline{r}_{\mu\nu_{0,0}} + F_3(\mu)F_4(\nu)\underline{r}_{\mu\nu_{0,1}} \\ & + F_3(\nu)F_4(\mu)\underline{r}_{\mu\nu_{1,0}} + F_4(\mu)F_4(\nu)\underline{r}_{\mu\nu_{1,1}} \qquad (12) \end{aligned}$$

where the blending functions can be taken as

$$\begin{aligned} F_1(\theta) &= 2\theta^3 - 3\theta^2 + 1 \\ F_2(\theta) &= 3\theta^2 - 2\theta^3 \\ F_3(\theta) &= \theta^3 - 2\theta^2 + \theta \\ F_4(\theta) &= \theta^3 - \theta^2 \end{aligned} \qquad (13)$$

and θ is a dummy variable. The mixed second derivatives of $\underline{r}$ were dropped from equation (12) in all calculations. As was discussed earlier in this chapter $\underline{r}$ and also the derivatives of $\underline{r}$ with respect to μ or ν can be calculated at each corner by finite differences. Thus, equations (14) can be used to find $\underline{r}$ at any point in any patch if the value of μ and ν for that point is known. Note that in equation (12) the two subscripts of $\underline{r}$, $\underline{r}_\mu$, and $\underline{r}_\nu$ refer to particular corners of a patch as shown in Figures 9 and 10.

As was stated earlier not only are the Cartesian coordinates $\underline{r} = (x,y,z)$ required for any point of a patch but also the (μ,ν) coordinates of that point are required. These can easily be found from equations (10) and (11) if we know which surface patch the point is in and the values of μ and ν for the patch.

Once the boundary points have been chosen and the blending equation (12) has been defined the differential equations (6) can be solved by some

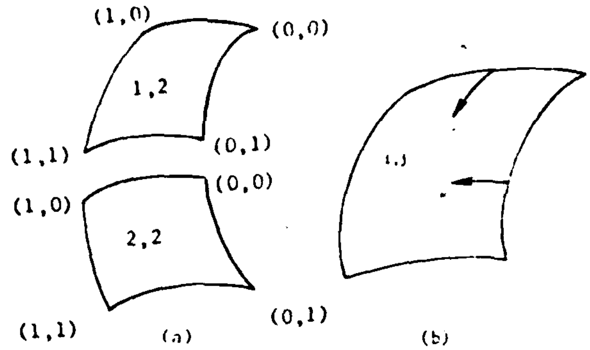


Fig. 9 (a) Two of the surface patches with labeling of corner points, (b) Isolated surface patch i,j showing parameters μ and ν.

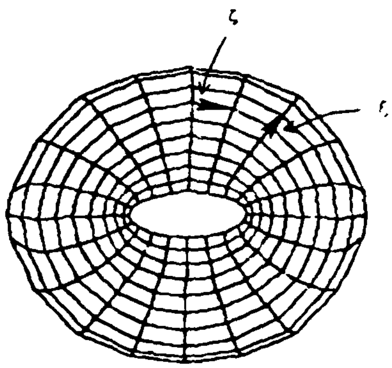


Fig. 10 (ζ,ξ) coordinate system on surface.

numerical scheme, in particular, point successive over relaxation (SOR) was used. The initial guess for the solution was taken as a simple linear interpolation between the values on the inner and outer curves. For the evaluation of the terms in the differential equations, i.e. to evaluate the coefficients a,b,c, J_2, $\Delta_2 u$, and $\Delta_2 v$ in (6) the blending equation which already has been defined is used.

RESULTS AND CONCLUSIONS

Results are shown for prolate ellipsoids of various thicknesses surrounded by either a sphere or an ellipsoid, Figs. (11) and (12). Also, considered as an inner body is a thin circular cross section fuselage, Fig. (13), intersecting spheres or ellipsoids, Figs. (14) and (15), and a wing-body combination. For the prolate ellipsoids and the fuselage results with and without coordinate concentration are shown.

The equations presented for generating a three-dimensional coordinate system between two arbitrary shaped bodies (or boundary surfaces) can be solved numerically and give good results for all cases attempted in this research. Inner body shapes which are simple (e.g. ellipsoids, fuselages, or intersecting spheres and ellipsoids) can easily be combined with an outer body to yield a very smooth coordinate system between them.

For the wing-body combination these same equations very easily adapt and generate a coordinate system as shown in Figures 16a and b. This application requires that the coordinates on the surface of the body be transformed by equations (6) to a system where one coordinate line is the trace of the wing and the body.

At this time the most imposing problem in the solution of equations (4) is the coorespondence of the points on the inner and outer boundary surfaces. As was discussed in reference to intersecting ellipsoids (Figures 2 and 3) this correspondence may dictate whether ornot the solution converges. The "gap" near the wing in Fig. 16 is due to the method used to correspond the points on the body to those on the outer boundary surface. Although conceptually it is not difficult to imagine various correspondences of the points on the two boundary surfaces it canbe quite difficult to generate this correspondence through a numerical algorithm

ACKNOWLEDGMENTS

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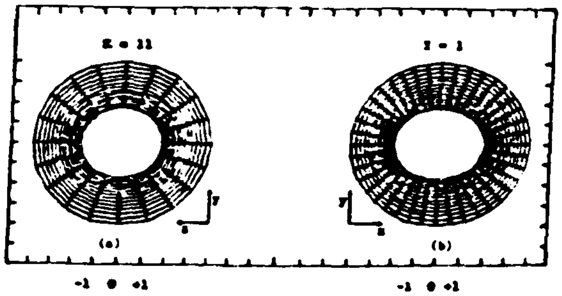


Fig. 11 Inner body a thick prolate ellipsoid with major axis 2 and minor axis $\sqrt{3}$ surrounded by a sphere of radius 4. (a) Coordinate contours for a section ζ = const (K = 11) for all (ξ,η) or (I,J) values; (b) for a section ξ = const (I = 1) for all (η,ζ) or (J,K) values. In both cases no contraction in , K = 1.

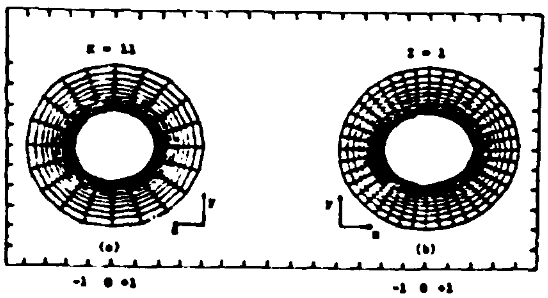
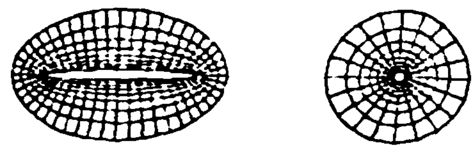
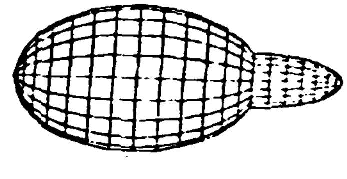


Fig. 12 Cases (a) and (b) of Fig. 11, with contraction in , K = 1.05.

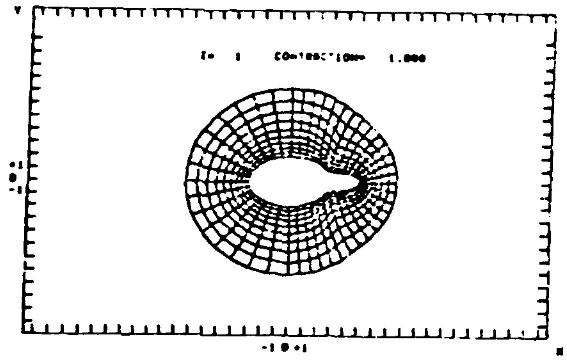


(a) (b)

Fig. 13 Inner body a thin prolate ellipsoid with major axis 2 and minor axis 0.13 surrounded by an ellipsoid with major axis 3 and minor axis 2.4. (a) Coordinate contours for a section ξ = constant (I = 1) for all (η,ζ) or (J,K) values; (b) for a section ζ = const (K = 13) for all (ξ,η) or (I,J) values. In both cases no contraction in η, K = 1.

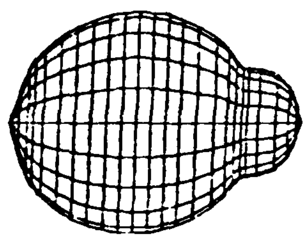


(a)

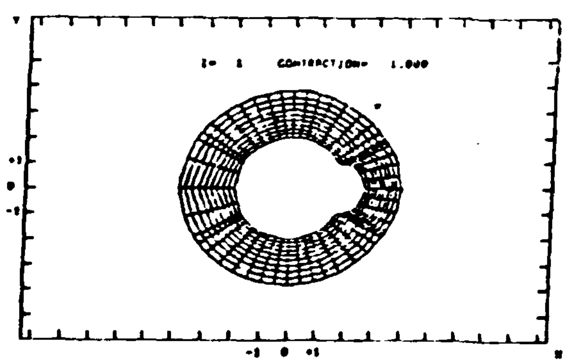


(b)

Fig. 15 (a) Coordinates on surface of two intersecting ellipsoids. (b) Coordinate contours between intersecting ellipsoids and a sphere for a section ξ = const (I = 1) for all (η,ζ) or (J,K) values. No contraction in η, K = 1.



(a)



(b)

Fig. 14 (a) Coordinates on surface of two intersecting spheres.
(b) Coordinate contours between intersecting spheres and an outer sphere for a section ξ = const (I = 1) for all (η,ζ) or (J,K) values. No contraction in η, K = 1.

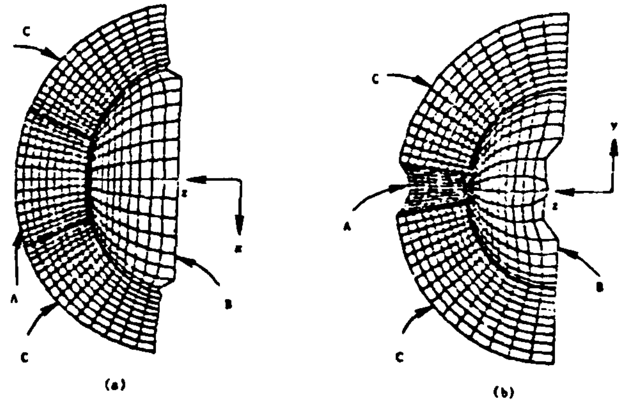


Fig. 16 Wing-body combination where wing is generated by extending lines from center of body through points on surface of body to outer shpere. $\underline{A}$ represents surface coordinates on wing. $\underline{B}$ represents surface coordinates on body. $\underline{C}$ represents three-dimensional coordinates between wing-body with x, y, & z coordinates. (a) top view of wing-body combination. (b) view from the front of wing-body combination.

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